"Around loop groups Langlands and mathematical physics"
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The categories $\mathcal{O}$ and Int

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$\mathfrak{g} = \mathring{\mathfrak{g}} \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$, the affine Lie algebra

has a triangular decomposition

$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ with $\quad \mathfrak{n}^+$ gen. by $e_0, e_1, \ldots, e_n$

$\mathfrak{h} = \mathring{\mathfrak{h}} \oplus \mathbb{C}K \oplus \mathbb{C}d$

$\mathfrak{n}^-$ gen by $f_0, f_1, \ldots, f_n$.

giving a decomposition of the enveloping algebra

$U = U\mathfrak{g} = U_{<0} U_0 U_{>0}$ where $U_{>0} = U\mathfrak{n}^+$,

$U_0 = U\mathfrak{h}$,

$U_{<0} = U\mathfrak{n}^-$.

The *category $\mathcal{O}$* is the category of $\mathfrak{g}$-modules M such that

(a) M is $\mathfrak{h}$-semisimple, i.e.

$$M = \bigoplus_{\gamma \in \mathfrak{h}^*} M_\gamma, \quad \text{where } M_\gamma = \{m \in M \mid hm = \gamma(h)m \text{ for } h \in \mathfrak{h}\}$$

(b) M is $\mathfrak{n}^+$-locally nilpotent, i.e.

~~if~~ $m \in M$ then $\dim(U_{>0}m) < \infty$,

(c) M is $U_{<0}$ finitely generated, i.e.

there exists $\ell \in \mathbb{Z}_{>0}$ and $v_1^+, \ldots, v_\ell^+ \in M$ with

$$M = U_{<0} v_1^+ + \cdots + U_{<0} v_\ell^+.$$

The *character* of M is

$$\mathrm{char}(M) = \sum_{\gamma \in \mathfrak{h}^*} \dim(M_\gamma) e^\gamma.$$

The category $\underline{\mathcal{I}nt}$ is the category of $\mathfrak{g}$-modules M s.t.

(a) M is $\mathfrak{h}$-semisimple,

(b) M is $\mathfrak{n}^+$-locally nilpotent,

(c) M is $\mathfrak{n}^-$-locally nilpotent.

<u>Remarks:</u>

(1) condition (a) is a consequence of (b) and (c) (see Kac Remark 3.6)

(2) $M \in \mathcal{I}nt$ are $\mathfrak{g}$-modules which "integrate" to G-modules,

$G = \exp(\mathfrak{g})$ acts on M by

$$e^{cX} \cdot m = \sum_{k \in \mathbb{Z}_{\geq 0}} \frac{c^k}{k!} X^k m, \quad \text{for } c \in \mathbb{C},\ X \in \mathfrak{g},\ m \in M.$$

Simple modules in $\mathcal{O}$ and Int

The Verma module of highest weight γ is

$$M = \mathcal{U} v_\gamma^+ = \mathcal{U}_{<0} v_\gamma^+ \quad \text{with} \quad \begin{array}{l} e_i v_\gamma^+ = 0, \text{ for } i = 0, 1, \ldots, n \\ h v_\gamma^+ = \gamma(h) v_\gamma^+, \text{ for } h \in \mathfrak{h}. \end{array}$$

The simple coroots $h_0, h_1, \ldots, h_n$ are

$h_i = [e_i, f_i]$. We also write $\alpha_i^\vee = h_i = [e_i, f_i]$.

The fundamental weights $\Lambda_0, \Lambda_1, \ldots, \Lambda_n$ and the element δ are the elements of $\mathfrak{h}^*$ given by

$$\Lambda_i(\alpha_j^\vee) = \delta_{ij}, \quad \Lambda_i(d) = 0, \quad \delta(\alpha_j^\vee) = 0, \quad \delta(d) = 1.$$

Let

$$\mathfrak{h}^*_{\mathbb{Z}_{\geq 0}} = \mathbb{Z}_{\geq 0}\Lambda_0 + \mathbb{Z}_{\geq 0}\Lambda_1 + \cdots + \mathbb{Z}_{\geq 0}\Lambda_n + \mathbb{C}\delta.$$

Theorem (a)

$$\left\{\begin{array}{c}\text{simple modules} \\ \text{in } \mathcal{O}\end{array}\right\} \longleftrightarrow \mathfrak{h}^*$$

$$L(\gamma) = \frac{M(\gamma)}{\left(\begin{array}{c}\text{max proper} \\ \text{submodule}\end{array}\right)} \longleftarrow\!\shortmid \gamma$$

(b)

$$\left\{\begin{array}{c}\text{simple modules} \\ \text{in } Int\end{array}\right\} \longleftrightarrow \mathfrak{h}^*_{\mathbb{Z}_{\geq 0}}.$$

$$L(\lambda) \longleftarrow\!\shortmid \lambda$$

The level of $L(\gamma)$ is $\gamma(K) = k$,

so that $K = k\, Id_{L(\gamma)}$, as operators on $L(\gamma)$.

<u>$W = W_0 \ltimes \mathring{\mathfrak{h}}_{\mathbb{Z}}$ acts on $\mathfrak{h}^* = \mathbb{C}\delta \oplus \mathring{\mathfrak{h}}^* \oplus \mathbb{C}\Lambda_0$</u>

$$\mathring{\mathfrak{h}}_{\mathbb{Z}} = \mathbb{Z}\text{-span}\{\alpha_1^\vee, \alpha_2^\vee, \ldots, \alpha_n^\vee\} \quad \text{and}$$

$$\mathring{\mathfrak{h}}_{\mathbb{Z}}^* = \mathbb{Z}\text{-span}\{\omega_1, \omega_2, \ldots, \omega_n\} \quad \text{where } \omega_i(\alpha_j^\vee) = \delta_{ij}.$$

The <u>finite Weyl group</u> W_0 is the subgroup of $GL(\mathring{\mathfrak{h}}_{\mathbb{Z}}^*)$,

$$W_0 \subseteq GL(\mathring{\mathfrak{h}}_{\mathbb{Z}}^*), \text{ generated by } s_1, s_2, \ldots, s_n$$

where

$$s_i : \mathring{\mathfrak{h}}_{\mathbb{Z}}^* \to \mathring{\mathfrak{h}}_{\mathbb{Z}}^* \quad \text{is given by} \quad s_i\lambda = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i.$$

The <u>affine Weyl group</u> is

$$W = W_0 \ltimes \mathring{\mathfrak{h}}_{\mathbb{Z}} = \{ w t_\beta \mid w \in W_0, \beta \in \mathring{\mathfrak{h}}_{\mathbb{Z}} \}$$

with

$$t_\beta t_\gamma = t_{\beta+\gamma} \quad \text{and} \quad w t_\beta = t_{w\beta} w,$$

acting on $\mathfrak{h}^*$ by

$$w = \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & W & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \quad \text{and} \quad t_\beta = \left(\begin{array}{c|c|c} 1 & -\beta^t & -\frac{1}{2}(\beta|\beta) \\ \hline 0 & 1 & \beta \\ \hline 0 & 0 & 1 \end{array}\right)$$

in the basis $\delta, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, \Lambda_0$ of $\mathfrak{h}^*$ where

$\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ is an orthonormal basis of $\mathring{\mathfrak{h}}^*$

w.r.t. a W_0-invariant symm. bilinear nondeg form $(\,|\,)$ on $\mathring{\mathfrak{h}}^*$.

$$\begin{aligned} \mathring{\mathfrak{h}}_{\mathbb{Z}} &\hookrightarrow \mathring{\mathfrak{h}}_{\mathbb{Z}}^* \\ \alpha_i^\vee &\mapsto \frac{2\alpha_i}{(\alpha_i|\alpha_i)} \end{aligned}$$

where $\alpha_i \in \mathfrak{h}^*$ is given by

$$[h, e_i] = \alpha_i(h) e_i, \text{ for } h \in \mathring{\mathfrak{h}}.$$

The algebra $\Theta[\mathring{\mathfrak{h}}^*_{\mathbb{Z}}]$

$$\Theta[\mathring{\mathfrak{h}}^*_{\mathbb{Z}}] = \mathbb{C}[\mathfrak{h}^*_{\mathbb{Z}}]^{\mathring{\mathfrak{h}}_{\mathbb{Z}}}$$

where $\mathfrak{h}^*_{\mathbb{Z}} = \mathbb{Z}\Lambda_0 + \cdots + \mathbb{Z}\Lambda_n + \mathbb{C}\delta$. Let

$$\Theta_\lambda = e^{-\frac{\langle\lambda,\lambda\rangle\delta}{2m}} \sum_{t_\beta \in \mathring{\mathfrak{h}}_{\mathbb{Z}}} e^{t_\beta(\lambda)}$$

For $a \in \mathbb{C}$ and $\beta \in \mathring{\mathfrak{h}}_{\mathbb{Z}}$,

$$\Theta_{a\delta+\bar{\lambda}+m\Lambda_0} = \Theta_{\bar{\lambda}+m\Lambda_0} \quad\text{and}\quad \Theta_{\bar{\lambda}+m\beta+m\Lambda_0} = \Theta_{\bar{\lambda}+m\Lambda_0}.$$

Let

$$G_1 = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) \in \mathbb{R}_{>0}\}, \qquad q = e^{2\pi i\tau} = e^{-\delta},$$

$$\Theta = \{\text{holomorphic functions } d : G_1 \to \mathbb{C}\}$$

Then

$$\Theta[\mathring{\mathfrak{h}}^*_{\mathbb{Z}}] = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \Theta[\mathring{\mathfrak{h}}^*_{\mathbb{Z}}]_m$$

is the <u>homogeneous</u> <u>coordinate</u> ring <u>of</u> $\mathring{\mathfrak{h}}^*_{\mathbb{C}} \big/ \left(\mathring{\mathfrak{h}}^*_{\mathbb{Z}} + \tau\mathring{\mathfrak{h}}_{\mathbb{Z}}\right)$,

$\Theta[\mathring{\mathfrak{h}}^*_{\mathbb{Z}}]_m$ has Θbasis $\{\Theta_{\bar{\lambda}+m\Lambda_0} \mid \bar{\lambda} \in \mathring{\mathfrak{h}}^*_{\mathbb{Z}} \bmod m\mathring{\mathfrak{h}}_{\mathbb{Z}}\}$

and

$$\Theta_{\bar{\lambda}+m\Lambda_0}\Theta_{\bar{\mu}+n\Lambda_0} = \sum_{?} d^{?}_{\bar{\lambda}+m\Lambda_0,\bar{\mu}+n\Lambda_0}\Theta_{?+(m+n)\Lambda_0}$$

with $d^{?}_{\bar{\lambda}+m\Lambda_0,\bar{\mu}+n\Lambda_0} \in \Theta$

Characters of simple modules $L(\lambda)$ in Int

Let
$$u_0 = \sum_{w\in W_0} w \qquad \text{and} \qquad e_0 = \sum_{w \in W_0} \det(w)\, w.$$

Let
$$\bar\rho = \omega_1 + \cdots + \omega_n \quad \text{and} \quad \rho = \Lambda_0 + \Lambda_1 + \cdots + \Lambda_n = \bar\rho + h^\vee \Lambda_0$$

The fundamental diagram for $\Theta[\mathring{\mathfrak{h}}^*_{\mathbb{Z}}]$ is

$$\begin{array}{ccc} u_0\,\Theta[\mathring{\mathfrak{h}}^*_{\mathbb{Z}}] & \xrightarrow{\sim} & e_0\,\Theta[\mathring{\mathfrak{h}}^*_{\mathbb{Z}}] \\ f & \longmapsto & A_\rho f \\ \chi_\lambda & \longleftarrow & A_{\lambda+\rho} = e_0\,\Theta_{\lambda+\rho} \end{array}$$

"not naive" basis $\qquad$ "naive basis"

__Theorem__ Let $L(\lambda) \in \mathrm{Int}$ be the irreducible $\mathfrak{g}$-module of highest weight λ.

(a) $\mathrm{char}(L(\lambda)) = e^{m_\lambda \delta} \chi_\lambda$, where $m_\lambda = \dfrac{(\lambda+\rho|\lambda+\rho)}{2(\lambda+\rho)(K)} - \dfrac{(\rho|\rho)}{2\rho(K)}$

(b) $A_\rho = e^{\bar\rho + h^\vee \Lambda_0} \prod_{k \in \mathbb{Z}_{>0}} (1-q^k)^n \prod_{\alpha \in \mathring{R}^+} (1-q^{k-1}e^{-\alpha})(1-q^k e^{\alpha})$

The action of $S=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ and $T=\begin{pmatrix}1&1\\0&1\end{pmatrix}$

Define an $SL_2(\mathbb{Z})$-action

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}(\tau,z,u)=\left(\frac{a\tau+b}{c\tau+d},\ \frac{z}{c\tau+d},\ u-\frac{c(z|z)}{2(c\tau+d)}\right)$$

so that

$$S\cdot(\tau,z,u)=\begin{pmatrix}0&-1\\1&0\end{pmatrix}(\tau,z,u)=\left(\frac{-1}{\tau},\ \frac{z}{\tau},\ u-\frac{(z|z)}{2\tau}\right)$$

$$T\cdot(\tau,z,u)=\begin{pmatrix}1&1\\0&1\end{pmatrix}(\tau,z,u)=(\tau+1,z,u)$$

Then

(a) For $\bar\lambda\in\mathring{\mathfrak{h}}^*_{\mathbb{Z}}$ mod $m\mathring{\mathfrak{h}}_{\mathbb{Z}}$,

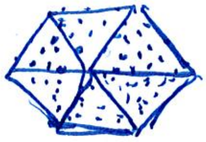
$$\Theta_{\bar\lambda+m\Lambda_0}(\tau+1,z,u)=e^{2\pi i\frac{(\bar\lambda,\bar\lambda)}{2m}}\Theta_{\bar\lambda+m\Lambda_0}(\tau,z,u)$$

(b) For $\bar\lambda\in\mathring{\mathfrak{h}}^*_{\mathbb{Z},\geq 0}$ mod $m\mathring{\mathfrak{h}}_{\mathbb{Z}}$

$$A_{\bar\lambda+m\Lambda_0}(\tau+1,z,u)=e^{2\pi i\frac{(\bar\lambda|\bar\lambda)}{2m}}A_{\bar\lambda+m\Lambda_0}(\tau,z,u)$$

(c) For $\bar\lambda\in\mathring{\mathfrak{h}}^*_{\mathbb{Z},\geq 0}$ mod $m\mathring{\mathfrak{h}}_{\mathbb{Z}}$

$$\chi_{\bar\lambda+m\Lambda_0}(\tau+1,z,u)=e^{2\pi i m_\lambda}\chi_{\bar\lambda+m\Lambda_0}(\tau,z,u)$$

(a) For $\bar{\lambda} \in \mathring{\mathfrak{h}}^*_{\mathbb{Z}} \bmod m\mathring{\mathfrak{h}}_{\mathbb{Z}}$, 

$$\Theta_{\bar{\lambda}+m\Lambda_0}\left(\frac{-1}{\tau}, \frac{z}{\tau}, u - \frac{(z|z)}{2\tau}\right) = \left(\frac{(-i\tau)^{\ell}}{d_m}\right)^{\frac{1}{2}} \sum_{\bar{\mu} \in \mathring{\mathfrak{h}}^*_{\mathbb{Z}} \bmod m\mathring{\mathfrak{h}}_{\mathbb{Z}}} e^{-\frac{2\pi i}{m}(\bar{\lambda}|\bar{\mu})} \Theta_{\bar{\mu}+m\Lambda_0}(\tau, z, u)$$

(b) For $\bar{\lambda} \in \mathring{\mathfrak{h}}^*_{\mathbb{Z}_{>0}} \bmod m\mathring{\mathfrak{h}}_{\mathbb{Z}}$ 

$$A_{\bar{\lambda}+m\Lambda_0}\left(\frac{-1}{\tau}, \frac{z}{\tau}, u - \frac{(z|z)}{2\tau}\right)$$

$$= \left(\frac{(-\tau i)^{\ell}}{d_m}\right)^{\frac{1}{2}} \sum_{\bar{\mu} \in \mathring{\mathfrak{h}}^*_{\mathbb{Z}_{>0}} \bmod m\mathring{\mathfrak{h}}_{\mathbb{Z}}} \left(\sum_{w \in W_0} \det(w)\, e^{-\frac{2\pi i}{m}(w\bar{\lambda}|\bar{\mu})} \right) A_{\bar{\mu}+m\Lambda_0}.$$

(c) For $\bar{\lambda} \in \mathring{\mathfrak{h}}^*_{\mathbb{Z}_{\geq 0}} \bmod m\mathring{\mathfrak{h}}_{\mathbb{Z}}$ 

$$\chi_{\bar{\lambda}+m\Lambda_0}\left(\frac{-1}{\tau}, \frac{z}{\tau}, u - \frac{(z|z)}{2\tau}\right) = \sum_{\bar{\mu} \in \mathring{\mathfrak{h}}^*_{\mathbb{Z}_{\geq 0}} \bmod m\mathring{\mathfrak{h}}_{\mathbb{Z}}} S_{\bar{\lambda}\bar{\mu}}\, \chi_{\bar{\mu}+m\Lambda_0}(\tau, z, u)$$

where

$$S_{\bar{\lambda},\bar{\mu}} = S_{\bar{\lambda},0}\, s_{\bar{\lambda}}\left(e^{\frac{2\pi i}{m+h^\vee} \nu^{-1}(\bar{\lambda}+\bar{\rho})}\right)$$

and

$$S_{\bar{\lambda},0} = \left(\frac{1}{d_{m+h^\vee}}\right)^{\frac{1}{2}} \prod_{\alpha \in \mathring{R}^+} 2 \sin\left(\frac{\pi(\bar{\lambda}+\bar{\rho}|\alpha)}{m+h^\vee}\right)$$

with

$$s_{\bar{\lambda}} = \mathrm{char}(\mathring{L}(\bar{\lambda}))$$

where $\mathring{L}(\bar{\lambda})$ is the simple $\mathring{\mathfrak{g}}$-module of highest weight $\bar{\lambda}$.