"Around loop groups, Langlands and Mathematical Physics"
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Virasoro modules

A. Ram

The Virasoro Lie algebra is the Lie algebra

$$Vir = \mathrm{span}\{\ldots, L_{-2}, L_{-1}, L_0, L_1, L_2, \ldots, c\}$$ with

$c \in Z(Vir)$ and $[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n}(m^3-m)\cdot c$

The triangular decomposition

$Vir = Vir^- \oplus Vir_0 \oplus Vir^+$ with
$$Vir^+ = \mathrm{span}\{L_1, L_2, \ldots\}$$
$$Vir_0 = \mathrm{span}\{L_0, c\}$$
$$Vir^- = \mathrm{span}\{L_{-1}, L_{-2}, \ldots\}.$$

gives a triangular decomp of the enveloping algebra $U = U(Vir)$,

$U = U_{>0} U_0 U_{<0}$ where
$$U_{>0} = U(Vir^+)$$
$$U_0 = U(Vir_0)$$
$$U_{<0} = U(Vir^-).$$

The Category $\mathcal{O}$ is the category of Vir-modules M with

(a) M is Vir_0-semisimple, i.e. $M = \bigoplus_{\gamma \in \mathfrak{h}^*} M_\gamma$ where
$$M_\gamma = \{m \in M \mid L_0 m = \gamma(L_0)m,\ cm = \gamma(c)m\}$$

(b) M is $U_{\geq 0}$-locally finite, i.e. if $m \in M$ then $\dim(U_{\geq 0} m) < \infty$

(c) M is $U_{\leq 0}$ finitely generated, i.e. there exists $\ell \in \mathbb{Z}_{>0}$ and $v_1^+, \ldots, v_\ell^+ \in M$ such that
$$M = U_{\leq 0} v_1^+ + \cdots + U_{\leq 0} v_\ell^+.$$

Verma modules and simple modules

Let $\mathfrak{h}^* = \text{span}\{\omega, \nu\}$ with $\omega(L_0)=1$, $\nu(L_0)=0$,
$\omega(c)=0$, $\nu(c)=1$.

Let $M \in \mathcal{O}$. The <u>character of M</u> is

$$\text{char}(M) = \sum_{\gamma\in\mathfrak{h}^*} \dim(M_\gamma)e^\gamma$$

Let $\gamma \in \mathfrak{h}^*$. The <u>Verma module of highest weight γ</u> is

$$M(\gamma) = \mathcal{U}v^+ = \mathcal{U}_{<0}v^+,$$ where $L_m v^+ = 0$ for $m \in \mathbb{Z}_{>0}$,

and $L_0 v^+ = \gamma(L_0)v^+$ and $cv^+ = \gamma(c)v^+$.

As $\mathfrak{h}$-modules,

$$\mathcal{U} = \bigoplus_{n\in\mathbb{Z}_{\geq 0}} \mathcal{U}_{-n},$$ where $\mathcal{U}_{-n} = \text{span}\{L_{-\lambda} \mid \lambda \vdash n\}$

where $\lambda \vdash n$ denotes $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ with $\lambda_1 \geq \cdots \geq \lambda_\ell$, $\lambda_1 + \cdots + \lambda_\ell = n$
and

$$L_{-\lambda} = L_{-\lambda_1}L_{-\lambda_2}\cdots L_{-\lambda_\ell}, \quad \text{for } \lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash n.$$

Let $q = e^\omega$ and $\kappa = e^\nu$. Then

$$\text{char}(M(\gamma)) = \kappa^{\gamma(c)} q^{\gamma(L_0)} \prod_{j=1}^{\infty} \frac{1}{1-q^j}$$

<u>Theorem</u> The simple modules in $\mathcal{O}$ are $L(\gamma)$, $\gamma \in \mathfrak{h}^*$,
with

$$L(\gamma) = \frac{M(\gamma)}{(\text{max. proper submodule})}$$

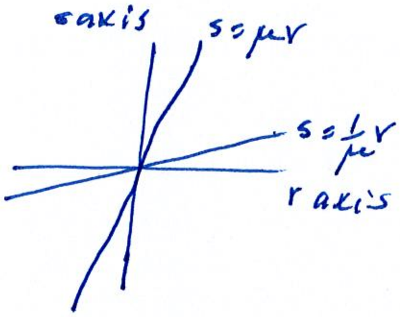
Blocks as W-orbits

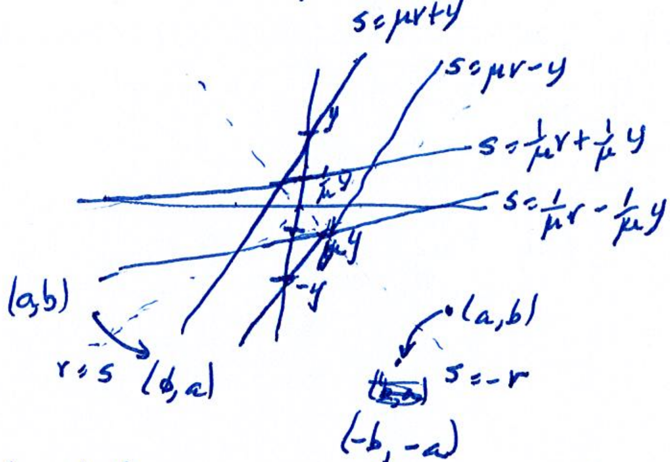
Let $\gamma = h\omega + c\Omega \in \mathfrak{h}^*$. Define ~~four~~ lines

$$s = \mu r + y, \quad s = \mu r - y, \quad s = \frac{1}{\mu} r - \frac{1}{\mu} y, \quad s = \frac{1}{\mu} r + \frac{1}{\mu} y$$

by

$$\mu + \frac{1}{\mu} = \frac{13-c}{6} \quad \text{and} \quad y^2 = 4\mu\left(\frac{1-c}{24} - h\right)$$





Given μ and y, $\frac{13-c}{6} = \mu + \frac{1}{\mu}$ determines c, and $h = \frac{-y^2}{4\mu} + \frac{1-c}{24}$ determines h.

Theorem

(a) If there is no integer point on the line $s = \frac{1}{\mu} r + \frac{1}{\mu} y$ then

$$M(\gamma) \text{ is simple.}$$

(b) If there is one integer point (a,b) on the line $s = \frac{1}{\mu} r + \frac{1}{\mu} y$ then

$$M(\gamma) \supseteq M(\gamma - ab\omega)$$

is a composition series of $M(\gamma)$

(c) If there is more than one integer point on $s = \frac{1}{\mu} r + \frac{1}{\mu} y$ then there are infinitely many integer points on $s = \frac{1}{\mu} r + \frac{1}{\mu} y$ ($\mu \in \mathbb{Q}$). Let $\mu = \frac{p}{q}$ with $\gcd(p,q)=1$ and $(a,b) \in \mathbb{Z}^2$ on $s = \frac{1}{\mu} r + \frac{1}{\mu} y$

The integer points on $s = \frac{1}{\mu} r + \frac{1}{\mu} y$ are

$(a + kq, b + kp)$, $k \in \mathbb{Z}$

Blocks / Inclusions of Verma modules

$M(h,c)$

$M(\gamma)$

$M(\gamma - ab\,\omega)$

$s = \mu r + y$ contains no integer points

$s = \mu r + y$ contains 1 integer point (a,b)

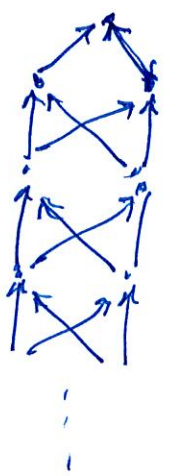
$M(\gamma)$

$M(\gamma - a_1 b_1 \omega)$

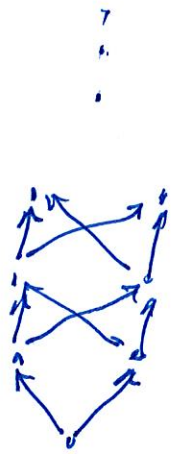
$M(\gamma - (a_1 b_1 + a_2 b_2)\omega)$

$s = \mu r + y$ with $\mu \in \mathbb{Q}_{>0}, y \in \mathbb{Z}$

$s = \mu y + r$ with $\mu \in \mathbb{Q}_{<0}, y \in \mathbb{Z}$



$s = \mu r + y$ with $\mu \in \mathbb{Q}_{>0}$
$y \notin \mathbb{Z}$



$s = \mu r + y$ with $\mu \in \mathbb{Q}_{<0}$
$y \notin \mathbb{Z}$.

The Sugawara and GKO constructions

$\mathring{\mathfrak{g}}$ a finite dimensional reductive Lie algebra
$\cup\!|$
$\mathring{\mathfrak{p}}$ a reductive Lie subalgebra

Let

$$\mathfrak{g} = \mathring{\mathfrak{g}} \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d \supseteq \mathfrak{p} = \mathring{\mathfrak{p}} \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$$
$$\cup\!| \qquad\qquad\qquad\qquad \cup\!|$$
$$\mathfrak{g}' = \mathring{\mathfrak{g}} \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}K \supseteq \mathfrak{p}' = \mathring{\mathfrak{p}} \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}K.$$

Let $\langle\, ,\rangle_0 : \mathring{\mathfrak{g}} \times \mathring{\mathfrak{g}} \to \mathbb{C}$ be a non. deg. ad. invariant symm. bilinear form.

Let

$h^\vee_{\mathfrak{g}}$ = dual Coxeter number for $\mathring{\mathfrak{g}}$,

$h^\vee_{\mathfrak{p}}$ = dual Coxeter number for $\mathring{\mathfrak{p}}$,

$\{J^a\}$ a basis of $\mathring{\mathfrak{g}}$ orthonormal w.r.t. $\langle\, ,\rangle_0$

$\{K^b\}$ a basis of $\mathring{\mathfrak{p}}$ orthonormal w.r.t. $\langle\, ,\rangle_0$.

For $X \in \mathring{\mathfrak{g}}$ let $X(z) = \sum_{n\in\mathbb{Z}} X(n) z^{-n-1}$ where $X(n) = Xt^n$

Define

$$T^{\mathfrak{g}}(z) = \frac{\dim(\mathring{\mathfrak{g}})}{h^\vee_{\mathfrak{g}}+l} \sum_{a=1}^{\dim \mathring{\mathfrak{g}}} :J^a(z) J^a(z): = \sum_{n\in\mathbb{Z}} L_n^{\mathfrak{g}} z^{-n-1}$$

$$T^{\mathfrak{p}}(z) = \frac{\dim(\mathring{\mathfrak{p}})}{h^\vee_{\mathfrak{p}}+l} \sum_{b=1}^{\dim(\mathring{\mathfrak{p}})} :K^b(z) K^b(z): = \sum_{n\in\mathbb{Z}} L_n^{\mathfrak{p}} z^{-n-1}$$

$$T(z) = T^{\mathfrak{g}}(z) - T^{\mathfrak{p}}(z) = \sum_{n\in\mathbb{Z}} (L_n^{\mathfrak{g}} - L_n^{\mathfrak{p}})\, z^{-n-1} = \sum_{n\in\mathbb{Z}} L_n z^{-n-1}.$$

A $\mathfrak{g}$-module V is <u>restricted</u> if V satisfies

if $v \in V$ then $\mathfrak{g}_\alpha v = 0$ for all but a finite number of roots α

A $\mathfrak{g}$ module V is <u>level ℓ</u> if K acts on V as $\ell \cdot Id_V$.

<u>Theorem</u> Let V be a restricted $\mathfrak{g}$-module of level ℓ. The L_n, $n \in \mathbb{Z}$, define an action of Vir on V with

c acting by $\left(\dfrac{\ell \dim(\mathring{\mathfrak{g}})}{h_{\mathfrak{g}}^{\vee}+\ell} - \dfrac{\ell \dim(\mathring{\mathfrak{p}})}{h_{\mathfrak{p}}^{\vee}+\ell}\right) \cdot Id_V$

This Vir action commutes with the $\mathfrak{p}$-action.

Unitarizable Vir-modules

A Vir-module V is *unitarizable* if there exists a pos. def. Hermitian form $\langle\, , \rangle : V\times V \to \mathbb{C}$ such that

(a) If $u, v\in V$ and $j\in\mathbb{Z}$ then $\langle L_j u, v\rangle = \langle u, L_{-j} v\rangle$,

(b) If $u, v\in V$ then $\langle cu, v\rangle = \langle u, cv\rangle$.

Theorem The simple module $L(h\omega + c\mathfrak{z})$ is unitarizable if and only if

(a) $h\in\mathbb{R}_{\geq 0}$ and $c\in\mathbb{R}_{\geq 0}$,

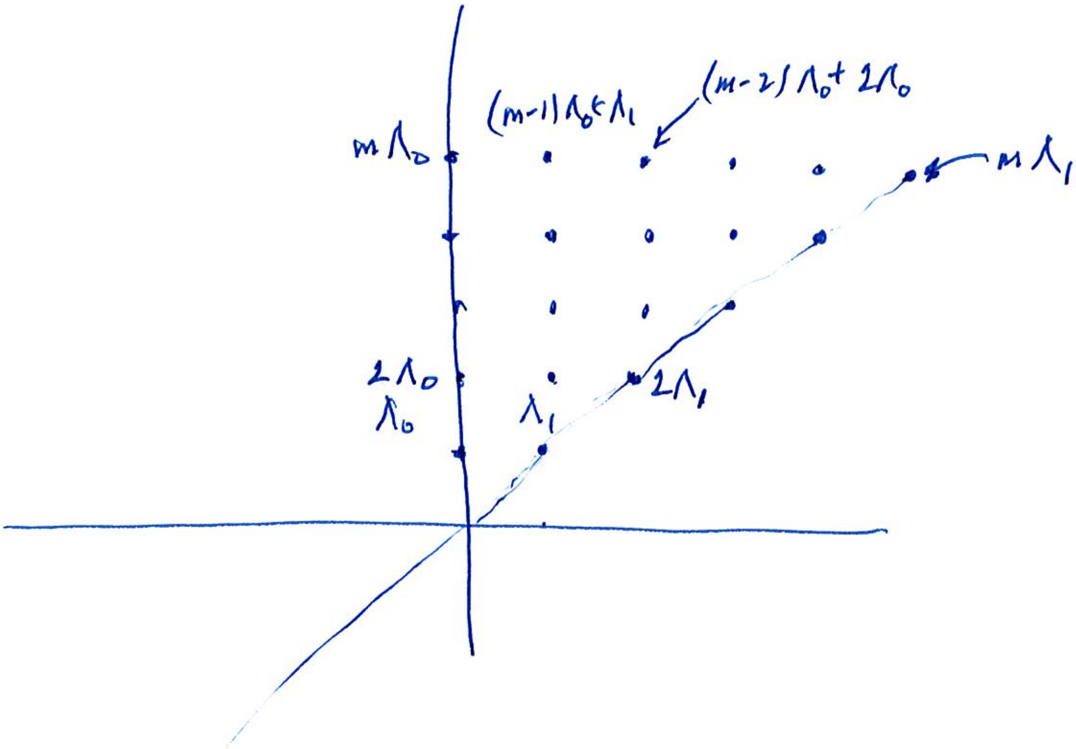
or (b) $$c = \frac{\binom{m+3}{2}-3}{\binom{m+3}{2}} \qquad h = \frac{((m+3)r-(m+2)s)^2-1}{4(m+2)(m+3)}$$

for $m\in\mathbb{Z}_{\geq 0}$, $r\in\{1,2,\ldots,m+1\}$, $s\in\{1,2,\ldots,m+2\}$.

Theorem Let Λ_0, Λ_1 be the fundamental weights of
$$\mathfrak{g}' = \mathfrak{sl}_2\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}K.$$
The GKO construction with $\mathring{\mathfrak{g}} = \mathfrak{sl}_2\oplus\mathfrak{sl}_2$ and $\mathring{\mathfrak{p}} = \mathfrak{sl}_2$ (diagonal embedding $\mathring{\mathfrak{p}} = \{(x,x) \mid x\in\mathfrak{sl}_2\}\subseteq\mathring{\mathfrak{g}}$) gives

$$L(\Lambda_0)\otimes L((m-j)\Lambda_0 + j\Lambda_1) = \sum_{\substack{0\leq k\leq m+1\\ j=k \bmod 2}}' L(h^{(m)}_{j+1,k+1}\omega + c^{(m)}\mathfrak{z})\otimes L((m+1-k)\Lambda_0 + k\Lambda_1)$$

as $\mathrm{Vir}\otimes\hat{\mathfrak{sl}}_2'$-modules.



$(m-1)\Lambda_0+\Lambda_1$
$(m-2)\Lambda_0+2\Lambda_0$
$m\Lambda_0$
$m\Lambda_1$
$2\Lambda_0$
Λ_0
Λ_1
$2\Lambda_1$