"Around loop groups, Langlands and mathematical physics,"

<u>What is a conformal field theory?</u> Lecture 4, <u>Conformal blocks</u> (1)

University of Melbourne
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$$\mathcal{M}_{g,N} = \left\{ \mathfrak{X} = (C; Q_1, \dots, Q_N) \;\middle|\; \begin{array}{l} C \text{ is an algebraic curve} \\ Q_1, Q_2, \dots, Q_N \in C \end{array} \right\}$$

(perhaps additional restrictions: C is semistable, C is nonsingular at $Q_1, \dots, Q_N$ etc. etc.)

For $\vec{\lambda} = (\lambda_1, \dots, \lambda_N) \in (P_\ell)^N$ define vector spaces $\mathcal{V}^\dagger_{\vec\lambda}(\mathfrak{X})$

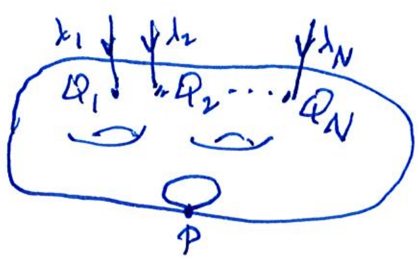
$\bigsqcup_{\mathfrak{X} \in \mathcal{M}_{g,N}} \mathcal{V}^\dagger_{\vec\lambda}(\mathfrak{X})$ is a vector bundle on $\mathcal{M}_{g,N}$

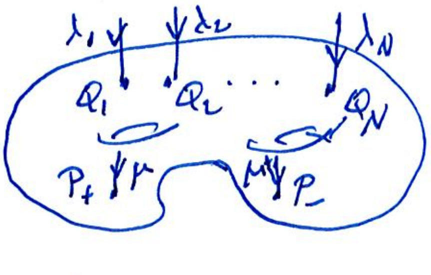
with a projectively flat connection $\nabla^{(\omega)}$

and, as N varies,

(C1) $\mathcal{V}^\dagger_{0,\vec\lambda}(C; P, Q_1, \dots, Q_N) \cong \mathcal{V}^\dagger_{\vec\lambda}(C; Q_1, \dots, Q_N)$

(C2) $\mathcal{V}^\dagger_{\vec\lambda}(C; Q_1, \dots, Q_N) \cong \bigoplus_{\mu \in P_\ell} \mathcal{V}^\dagger_{\vec\lambda, \mu, \mu^\dagger}(\hat{C}; Q_1, Q_2, \dots, Q_N, P_+, P_-)$





((C3) is a condition that all fibers ~~have~~ of $\mathcal{V}^\dagger_{\vec\lambda}(\cdot)$ the same dimension)

The spaces $\mathcal{H}_{\vec\lambda}$ and $\mathcal{H}_{\vec\lambda}^{+}$

Let $\mathring{\mathfrak{g}}$ be a finite dimensional complex Lie algebra with

bracket $[\,,\,]_0 : \mathring{\mathfrak{g}} \otimes \mathring{\mathfrak{g}} \to \mathring{\mathfrak{g}}$ and $\langle\,,\,\rangle_0 : \mathring{\mathfrak{g}} \times \mathring{\mathfrak{g}} \to \mathbb{C}$

a nondegenerate symmetric ad-invariant bilinear form.

Let

$$\mathfrak{g}_N = \mathring{\mathfrak{g}} \otimes \mathbb{C}((\xi_1)) \oplus \cdots \oplus \mathring{\mathfrak{g}} \otimes \mathbb{C}((\xi_N)) \oplus \mathbb{C}K$$

with bracket $K \in Z(\mathfrak{g}_N)$ and

$$[X_1\xi_1^{m_1} + \cdots + X_N\xi_N^{m_N}, Y_1\xi_1^{n_1} + \cdots + Y_N\xi_N^{n_N}]$$
$$= \sum_{j=1}^{N} [X_j, Y_j]_0\, \xi_j^{m_j+n_j} + K \sum_{j=1}^{N} \delta_{m_j,-n_j}\, m_j \langle X_j, Y_j\rangle_0$$

$\mathfrak{g}_1$ is the affine Lie algebra corresponding to $\mathring{\mathfrak{g}}$.

P_ℓ is an index set for

$\mathcal{H}_\lambda$, the irred. integrable $\mathfrak{g}_1$-modules of level ℓ.

$\mathcal{H}_\lambda^{+}$ is the (graded) dual of $\mathcal{H}_\lambda$.

For $\vec\lambda = (\lambda_1, \ldots, \lambda_N)$ with $\lambda_1, \ldots, \lambda_N \in P_\ell$ define

$$\mathcal{H}_{\vec\lambda} = \mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_N} \quad \text{and} \quad \mathcal{H}_{\vec\lambda}^{+} = \mathcal{H}_{\lambda_1}^{+} \otimes \cdots \otimes \mathcal{H}_{\lambda_N}^{+}$$

with $\mathfrak{g}_N$-action given by

$$(X_1 f_1(\xi_1) + \cdots + X_N f_N(\xi_N))(v_1 \otimes \cdots \otimes v_N)$$
$$= \sum_{j=1}^{N} v_1 \otimes \cdots \otimes v_{j-1} \otimes X_j f_j(\xi_j) v_j \otimes v_{j+1} \otimes \cdots \otimes v_N$$

The spaces $\mathcal{V}_{\vec{\lambda}}$ and $\mathcal{V}_{\vec{\lambda}}^{\dagger}$

Let $\mathfrak{X} = (C; Q_1, \ldots, Q_N; s_1, s_2, \ldots s_N)$ be a stable N-pointed curve with formal neighborhoods. Define

$$\Gamma\mathcal{O}(\mathfrak{X}) = H^0(C, \mathcal{O}_C(*\sum_{j=1}^{N} Q_j)) \hookrightarrow \bigoplus_{j=1}^{N} \mathbb{C}((\xi_j))$$

$$\Gamma\omega(\mathfrak{X}) = H^0(C, \omega_C(*\sum_{j=1}^{N} Q_j)) \hookrightarrow \bigoplus_{j=1}^{N} \mathbb{C}((\xi_j))\, d\xi_j .$$

and let

$$\mathfrak{g}(\mathfrak{X}) = \mathring{\mathfrak{g}} \otimes \Gamma\mathcal{O}(\mathfrak{X}) \hookrightarrow \left(\bigoplus_{j=1}^{N} \mathring{\mathfrak{g}} \otimes \mathbb{C}((\xi_j))\right) \oplus \mathbb{C}K$$

Define

$$\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{X}) = \mathrm{Hom}_{\mathbb{C}}\left(\frac{\mathcal{H}_{\vec{\lambda}}}{\mathfrak{g}(\mathfrak{X})\mathcal{H}_{\vec{\lambda}}}, \mathbb{C}\right),$$
the vector space of <u>conformal blocks of $\mathfrak{X}$ with label $\vec{\lambda}$</u>

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}) = \frac{\mathcal{H}_{\vec{\lambda}}}{\mathfrak{g}(\mathfrak{X})\mathcal{H}_{\vec{\lambda}}},$$
the vector space of <u>dual conformal blocks of $\mathfrak{X}$ with label $\vec{\lambda}$</u>

Note that

$$\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{X}) = \{ \langle \Psi | \in \mathcal{H}_{\vec{\lambda}}^{\dagger} \mid \underbrace{\langle \Psi | \mathfrak{g}(\mathfrak{X}) = 0}_{\text{gauge condition.}} \}$$

The spaces $\mathcal{V}_{\vec\lambda}(\mathcal{F})$ and $\mathcal{V}^{\dagger}_{\vec\lambda}(\mathcal{F})$

For a family of curves of genus g with N marked points with formal neighborhoods

$$\mathcal{F} = (\mathcal{C} \to \mathcal{B};\ s_1, \ldots, s_N;\ \eta_1, \ldots, \eta_N)$$

Let

$$\Gamma\mathcal{O}(\mathcal{B}) = \pi_*\Big(\mathcal{O}_{\mathcal{C}}\big(* \sum_{j=1}^{N} s_j(\mathcal{B})\big)\Big) \hookrightarrow \bigoplus_{j=1}^{N} \mathcal{O}_{\mathcal{B}}((\xi_j))$$

$$\mathfrak{g}(\mathcal{F}) = \mathring{\mathfrak{g}} \otimes \Gamma\mathcal{O}(\mathcal{B}) \subseteq \Big(\bigoplus_{j=1}^{N} \mathring{\mathfrak{g}} \otimes \mathcal{O}_{\mathcal{B}}((\xi_j))\Big) \oplus \mathcal{O}_{\mathcal{B}} \cdot K.$$

$$\mathcal{H}_{\vec\lambda}(\mathcal{B}) = \mathcal{O}_{\mathcal{B}} \otimes \mathcal{H}_{\vec\lambda} \quad \text{and} \quad \mathcal{H}^{\dagger}_{\vec\lambda}(\mathcal{B}) = \underline{\mathrm{Hom}}_{\mathcal{O}_{\mathcal{B}}}(\mathcal{H}_{\vec\lambda}(\mathcal{B}), \mathcal{O}_{\mathcal{B}}).$$

Define

$$\mathcal{V}_{\vec\lambda}(\mathcal{F}) = \frac{\mathcal{H}_{\vec\lambda}(\mathcal{B})}{\mathfrak{g}(\mathcal{F})\mathcal{H}_{\vec\lambda}(\mathcal{B})},$$ the <u>sheaf of dual conformal blocks associated to $\mathcal{F}$</u>

$$\mathcal{V}^{\dagger}_{\vec\lambda}(\mathcal{F}) = \underline{\mathrm{Hom}}_{\mathcal{O}_{\mathcal{B}}}(\mathcal{V}_{\vec\lambda}(\mathcal{F}), \mathcal{O}_{\mathcal{B}}),$$ the <u>sheaf of conformal blocks associated to $\mathcal{F}$</u>

(The favorite example of a family of $\mathbb{P}^1$ is

$$\mathcal{B} = \mathbb{C}^N - \Big(\bigcup_{1 \le i < j \le N} H_{ij}\Big) \quad \text{with } H_{ij} = \{(z_1, \ldots, z_N) \in \mathbb{C}^N \mid z_i = z_j\}$$

$$\mathcal{C} = \mathcal{B} \times \mathbb{P}^1$$

$$s_j : \mathcal{B} \longrightarrow \mathcal{B} \times \mathbb{P}^1$$
$$(z_1, \ldots, z_N) \longmapsto (z_1, \ldots, z_N; [z_j; 1])$$
)

The Vir action on $\mathcal{H}_\lambda$

The Virasoro algebra is the Lie algebra

$$Vir = \mathrm{span}\{\dots, L_{-2}, L_{-1}, L_0, L_1, L_2, \dots, c\} \text{ with}$$

$c \in Z(Vir)$ and $[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n}\frac{m^3-m}{12}c$.

The favorite Vir module is $\mathbb{C}((\xi))$ with $L_m = -\xi^{m+1}\frac{d}{d\xi}$

~~Let~~

$$\mathfrak{g} = \mathfrak{g}_1 = \mathring{\mathfrak{g}}[\xi, \xi^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$$

is the affine Lie algebra corresponding to $\mathring{\mathfrak{g}}$.

For $X \in \mathring{\mathfrak{g}}$, the current defined by X is

$$X(z) = \sum_{n \in \mathbb{Z}} X(n) z^{-n-1}, \quad \text{where } X(n) = X\xi^n.$$

Let $\{J^a\}$ be an orthonormal basis of $\mathring{\mathfrak{g}}$ w.r.t. $\langle , \rangle_0$.

Let $\Lambda_0, \Lambda_1, \dots, \Lambda_n$ be the fundamental weights of $\mathfrak{g}$ and

$$g^* = h^\vee = \rho(K) = (\Lambda_0 + \Lambda_1 + \dots + \Lambda_n)(K) \text{ the dual Coxeter number.}$$

The energy-momentum tensor is

$$T(z) = \frac{1}{2(g^*+\ell)} \sum_{a=1}^{\dim \mathring{\mathfrak{g}}} :J^a(z) J^a(z): = \sum_{m \in \mathbb{Z}} L_m z^{-m-2}$$

Theorem Let $\mathcal{H}_\lambda$ be an integrable $\mathfrak{g}$-module of level ℓ.
The L_m, $m \in \mathbb{Z}$, provide an action of Vir on $\mathcal{H}_\lambda$ with c acting by $\frac{\ell \dim \mathring{\mathfrak{g}}}{g^*+\ell}$ (the conformal anomaly).

The $\mathcal{L}(\mathcal{F})$ action on $\mathcal{V}_{\vec{\lambda}}(\mathcal{F})$

For a family of curves of genus g with N marked points with formal neighborhoods

$$\mathcal{F} = (\mathcal{C} \xrightarrow{\pi} \mathcal{B};\, s_1, \dots, s_N;\, \eta_1, \dots, \eta_N)$$

Let

$$\mathcal{L}(\mathcal{F}) = \mathcal{O}_{\mathcal{B}}[\xi_1^{-1}]\frac{d}{d\xi_1} \oplus \cdots \oplus \mathcal{O}_{\mathcal{B}}[\xi_N^{-1}]\frac{d}{d\xi_N}.$$

Then $\vec{\ell} = \ell_1 \frac{d}{d\xi_1} + \cdots + \ell_N \frac{d}{d\xi_N} \in \mathcal{L}(\mathcal{F})$ acts on $\mathcal{H}_{\lambda}(\mathcal{B})$ and on $\mathcal{V}_{\vec{\lambda}}(\mathcal{F})$, by the operator $D(\vec{\ell})$:

for $f \in \mathcal{O}_{\mathcal{B}}$ and $v_1 \in \mathcal{H}_{\lambda_1}, \dots, v_N \in \mathcal{H}_{\lambda_N}$,

$$D(\vec{\ell})(f(v_1 \otimes \cdots \otimes v_N))$$

$$= \theta(\vec{\ell})f \cdot (v_1 \otimes \cdots \otimes v_N) - f\left(\sum_{j=1}^{N} v_1 \otimes \cdots \otimes v_{j-1} \otimes T[\underline{\ell}_j] v_j \otimes v_{j+1} \otimes \cdots \otimes v_N\right)$$

where $T[\underline{\ell}_j]$ is the linear extension of

$$T\left[\xi_j^{-m} \frac{d}{d\xi_j}\right] = \delta_{-n-2-m=-1} L_n = L_{-m-1}$$

and $\theta(\vec{\ell})$ is the derivation of $\mathcal{O}_{\mathcal{B}}$ corresponding to $\vec{\ell}$ by pulling back across $s_j : \mathcal{B} \to \mathcal{C}$.

The connection $\nabla^{(\omega)}$ on $\mathcal{V}_{\vec{\lambda}}(\mathcal{F})$

Let $\omega \in H^0(\mathcal{C} \times_{\mathcal{B}} \mathcal{C}, \omega_{\mathcal{C}\times_{\mathcal{B}}\mathcal{C}/\mathcal{B}}(2\Delta))$ such that locally $\omega \in H^0(C \times C, \Omega^2_{C\times C}(2\Delta))$ is a meromorphic 2-form and, around the diagonal Δ,

$$\omega(z,w) = \frac{1}{(z-w)^2}\, dz \wedge dw + \text{regular}.$$

Define

$$S_\omega(z)(dz)^2 = 6 \lim_{w \to z} \left(\omega(w,z)\, dw\, dz - \frac{dw\, dz}{(w-z)^2} \right)$$

For $\vec{l} = l_1(\xi_1)\frac{d}{d\xi_1} + \cdots + l_N(\xi_N)\frac{d}{d\xi_N}$ define $a_\omega(\vec{l}) \in \mathcal{O}_{\mathcal{B}}$ by

$$a_\omega(\vec{l}) = \frac{-1}{12}\left(\frac{\ell \dim \mathfrak{g}}{g^* + \ell}\right) \sum_{j=1}^{N} \operatorname*{Res}_{\xi_j = 0} \left(l_j(\xi_j) S_\omega(\xi_j)\, d\xi_j \right)$$

(Note: If $C = \mathbb{P}^1$ then $a_\omega = 0$, since $\omega = \frac{dw\, dz}{(w-z)^2}$ is a global diff form)

For $X \in \Theta_{\mathcal{B}}(-\log \mathcal{D})$ choose $\vec{l} \in \mathcal{L}(\mathcal{F})$ with $\theta(\vec{l}) = X$, and define

$$\nabla^{(\omega)}_X \in \operatorname{End}(\mathcal{V}_{\vec{\lambda}}(\mathcal{F})) \text{ by } \nabla^{(\omega)}_X = D(\vec{l}) - a_\omega(\vec{l})$$

<u>Then</u> $\nabla^{(\omega)}_X$ defines a projectively flat connection on $\mathcal{V}_{\vec{\lambda}}(\mathcal{F})$.