Twisted Chevalley groups

$G = G(\mathbb{F})$ a Chevalley group (the $\mathbb{F}$-points of the group scheme for a complex reductive alg. group).

$\sigma: G \to G$ an automorphism (order m, $\sigma^m = 1$).

with

$$\sigma(U) = U, \quad \sigma(U^-) = U^-, \quad \sigma(H) = H, \quad \sigma(N) = N.$$

The twisted Chevalley group is

$$G^\sigma = \{g \in G \mid \sigma(g) = g\}.$$

If $G = GL_n(\mathbb{F})$ then

$$U = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in G \right\}, \quad U^- = \left\{ \begin{pmatrix} 1 & & \\ & \ddots & \\ * & & 1 \end{pmatrix} \in G \right\}$$

$$H = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \in G \right\}, \quad N = \left\{ g \in G \;\middle|\; \begin{matrix} g \text{ has exactly one} \\ \text{nonzero entry in} \\ \text{each row and each col} \end{matrix} \right\}$$

Steinberg Theorem 30 Let $G(\mathbb{F})$ be a Chevalley group. Assume $F: \mathbb{F} \to \mathbb{F}$, $x \mapsto x^p$ ($p = \mathrm{char}(\mathbb{F})$) is an automorphism and the root system is indecomposable. Every automorphism is the product of an inner, a diagonal, a graph, and a field automorphism.

Twisted affine Lie algebras

$\mathring{\mathfrak{g}}$ a finite dimensional complex simple Lie algebra.

$\sigma : \mathring{\mathfrak{g}} \to \mathring{\mathfrak{g}}$ an automorphism (order m, $\sigma^m = 1$).

Kac Proposition 8.1 There exists a Cartan subalgebra $\mathring{\mathfrak{h}}$ of $\mathring{\mathfrak{g}}$ such that

$$\sigma \text{ is conjugate to } \mu \exp\left(\frac{2\pi i}{m} h\right)$$

with $h \in \mathring{\mathfrak{h}}$ and μ a diagram automorphism.

Let $\mathfrak{g}$ be the affine Kac-Moody algebra of $\mathring{\mathfrak{g}}$,

$$\mathfrak{g} = \mathring{\mathfrak{g}}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$$

Define an automorphism $\tilde{\sigma} : \mathfrak{g} \to \mathfrak{g}$ by

$$\tilde{\sigma}(xt^j) = e^{-2\pi i j/m} \sigma(x) t^j, \quad \tilde{\sigma}(K) = K, \quad \tilde{\sigma}(d) = d,$$

The twisted affine Kac-Moody algebra is

$$\mathfrak{g}^{\tilde{\sigma}} = \{ y \in \mathfrak{g} \mid \tilde{\sigma}(y) = y \}.$$

The Lie algebra $\mathring{\mathfrak{g}}$ has $\mathbb{Z}/m\mathbb{Z}$-grading

$$\mathring{\mathfrak{g}} = \bigoplus_{j=0}^{m-1} \mathring{\mathfrak{g}}_j, \quad \text{where } \mathring{\mathfrak{g}}_j = \{ x \in \mathring{\mathfrak{g}} \mid \sigma(x) = e^{2\pi i j/m} x \}$$

Then

$$\mathfrak{g}^{\tilde{\sigma}} = \left(\bigoplus_{j \in \mathbb{Z}} t^j \mathring{\mathfrak{g}}_{j \bmod m} \right) \oplus \mathbb{C}K \oplus \mathbb{C}d.$$

The unitary group $U_n(\mathbb{F})$ (type ${}^2A_n$)

Let $\theta : \mathbb{F} \to \mathbb{F}$ be a field automorphism with $\theta^2 = 1$.

Let $G = GL_n(\mathbb{F})$,

$$a = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \quad \text{and} \quad \begin{aligned} \sigma : GL(\mathbb{F}) &\longrightarrow GL_n(\mathbb{F}) \\ g &\longmapsto a(\bar{g}^t)^{-1} a^{-1} \end{aligned}$$

where $\bar{g} = (\theta(g_{ij}))$. Then

$$\begin{aligned} GL_n(\mathbb{F})^\sigma &= \{ g \in GL_n(\mathbb{F}) \mid g a \bar{g}^t = a \} \\ &= \{ g \in GL_n(\mathbb{F}) \mid \langle gx, gy \rangle = \langle x, y \rangle \} \end{aligned}$$

where the Hermitian/sesquilinear form $\langle , \rangle : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}$ is given by $\langle x, y \rangle = x^t a \bar{y}$.

Then

$$\begin{aligned} GL_n(\mathbb{F})^\sigma &\xrightarrow{\sim} U_n(\mathbb{F}) \\ g &\longmapsto ga \end{aligned}$$

where

$$U_n(\mathbb{F}) = \{ h \in GL_n(\mathbb{F}) \mid h \bar{h}^t = 1 \}.$$

Root subgroups and affine root systems

A Chevalley group $\mathring{G}(\mathbb{F})$ "comes with"

generators $x_\alpha(c)$ for $\alpha \in \mathring{R}$, $c \in \mathbb{F}$.

which are analogues of elementary matrices.

$$x_\alpha(c_1)x_\alpha(c_2) = x_\alpha(c_1+c_2)$$

so that the <u>root</u> <u>subgroup</u>

$$\mathcal{X}_\alpha = \{x_\alpha(c) \mid c \in \mathbb{F}\} \cong \mathbb{F}^+$$

Also the subgroup generated by $\mathcal{X}_\alpha$ and $\mathcal{X}_{-\alpha}$,

$$\langle \mathcal{X}_\alpha, \mathcal{X}_{-\alpha} \rangle \cong SL_2(\mathbb{F}).$$

Write

$\mathring{G}(\mathbb{C}((t)))$ is the <u>loop group</u>

$\mathring{G}(\mathbb{Q}_p)$ is the <u>p-adic group</u>

$\mathbb{C}((t))$ and $\mathbb{Q}_p$ are <u>fields</u> <u>with</u> <u>valuation</u> and the <u>affine</u> <u>root</u> <u>subgroups</u> in $\mathring{G}(\mathbb{C}((t)))$ are

$$\mathcal{X}_{\alpha+j\delta} = \{x_\alpha(ct^j) \mid c \in \mathbb{C}\} \text{ for } \alpha \in \mathring{R},\ j \in \mathbb{Z}.$$

This is the source of <u>affine</u> <u>root</u> <u>systems</u>.

Root subgroups for $GL_n(\mathbb{F})$

Let

$$\mathfrak{h}_{\mathbb{Z}}^* = \mathbb{Z}\text{-span}\{\varepsilon_1, \ldots, \varepsilon_n\} \quad \text{and}$$

$$\begin{aligned}\mathring{R} &= \{\pm(\varepsilon_i - \varepsilon_j) \mid 1 \le i < j \le n\} \\ &= \{\varepsilon_k - \varepsilon_\ell \mid k, \ell \in \{1, \ldots, n\} \text{ and } k \ne \ell\}\end{aligned}$$

Then $GL_n(\mathbb{F})$ is generated by

$$x_{\varepsilon_k - \varepsilon_\ell}(c) \quad \text{for } \varepsilon_k - \varepsilon_\ell \in \mathring{R} \text{ and } c \in \mathbb{F}$$

and $h_i(d)$ for $i = 1, 2, \ldots, n$ and $d \in \mathbb{F}^\times$,

$$x_{\varepsilon_k - \varepsilon_\ell}(c) = \begin{array}{c} \\ k^{th} \end{array}\!\!\begin{array}{c} \quad\ \ell^{th} \\ \begin{pmatrix} 1 & & c \\ & \ddots & \\ & & 1 \end{pmatrix} \end{array} \quad \text{and} \quad h_i(d) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & d & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

The affine root system for $GL_n(\mathbb{C}((t)))$ has

$$x_{\varepsilon_k - \varepsilon_j + j\delta}(c) = x_{\varepsilon_k - \varepsilon_\ell}(ct^j) = \begin{array}{c} \\ k^{th} \end{array}\!\!\begin{array}{c} \quad\ \ell^{th} \\ \begin{pmatrix} 1 & & ct^j \\ & \ddots & \\ & & 1 \end{pmatrix} \end{array}$$