"Around loop groups, Langlands and mathematical physics"
Univ. of Melbourne, Lecture 2: Central extensions: Heisenberg, Virasoro and affine Lie algebras, 4 March 2015 (1)

A. Ram

Central extensions for groups

G a group (with operation $\circ$)

C an (abelian) group.

A <u>central</u> <u>extension</u> of G by C is a group

$$CG = \{cg \mid c \in C, g \in G\} \quad \text{with}$$

C a subgroup of CG contained in $Z(CG)$, and

G is <u>not</u> a subgroup of CG,

$$g_1 g_2 = f(g_1, g_2)(g_1 \circ g_2), \quad \text{for } g_1, g_2 \in G,$$

where $f: G \times G \to C$ and associativity in CG forces

$$f(g_1, g_2) f(g_1 \circ g_2, g_3) = f(g_2, g_3) f(g_1, g_2 \circ g_3)$$

so that f is a 2-cocycle on G with values in C.

$$\{1\} \longrightarrow C \longrightarrow CG \longrightarrow G \longrightarrow \{1\}.$$

Central extensions for Lie algebras

$\mathfrak{g}$ a Lie algebra (with bracket $[\,,\,]_0$)

$\mathfrak{c}$ an abelian Lie algebra ($[c_1, c_2] = 0$ for $c_1, c_2 \in \mathfrak{c}$).

A central extension of $\mathfrak{g}$ by $\mathfrak{c}$ is a Lie algebra

$$\hat{\mathfrak{g}} = \mathfrak{c} \oplus \mathfrak{g} \quad \text{with}$$

$\mathfrak{c}$ a Lie subalgebra of $\hat{\mathfrak{g}}$ contained in $Z(\hat{\mathfrak{g}})$,

$\mathfrak{g}$ is not a Lie subalgebra of $\hat{\mathfrak{g}}$,

$$[X_1, X_2] = f(X_1, X_2) + [X_1, X_2]_0, \quad \text{for } X_1, X_2 \in \mathfrak{g},$$

where $f: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{c}$ and the Jacobi identity forces

$$f([X_1, X_2]_0, X_3) + f([X_2, X_3]_0, X_1) = f([X_1, X_3]_0, X_2)$$

so that f is a 2-cocycle on $\mathfrak{g}$ with values in $\mathfrak{c}$.

$$\{0\} \longrightarrow \mathfrak{c} \longrightarrow \hat{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow \{0\}.$$

Lie algebras and Lie groups

There is a functor

$$\{\text{Lie algebras}\} \longrightarrow \{\text{Lie groups}\}$$

$$\mathfrak{g} \longmapsto G = \langle \exp(tX) \mid t \in \mathbb{R},\ X \in \mathfrak{g} \rangle$$

by the Baker-Campbell-Hausdorff formula

$$e^{tX} e^{sY} = e^{tX + sY + \frac{1}{2} st[X,Y] + \cdots}$$

<u>HW</u>: What about a functor

$$\{\text{Lie groups}\} \longrightarrow \{\text{Lie algebras}\}$$

$$G \longmapsto \mathfrak{g} = T_1 G$$

What restrictions make these functors provide equivalences of categories?

The Heisenberg group

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, x, y, z \in \mathbb{R} \right\} = CG$$

$$C = Z(H) = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, z \in \mathbb{R} \right\}$$

$G = \{(x,y) \mid x, y \in \mathbb{R}\}$ with $(x_1,y_1)+(x_2,y_2) = (x_1+x_2, y_1+y_2)$.

Then

$$\{1\} \longrightarrow C \longrightarrow H \longrightarrow G \longrightarrow \{1\}$$

$$\begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \longmapsto (x,y)$$

and

$$\begin{pmatrix} 1 & x_1 & 0 \\ & 1 & y_1 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & x_2 & 0 \\ & 1 & y_2 \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & x_1 y_2 \\ & 1 & 0 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & x_1+x_2 & 0 \\ & 1 & y_1+y_2 \\ & & 1 \end{pmatrix}$$

so that

$f : G \times G \to C$ is $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by

$$f((x_1,y_1),(x_2,y_2)) = x_1 y_2$$

The Weyl algebra

$$\mathcal{D} = \mathrm{span}\{p, q, h\} \quad \text{with}$$

$$[p, q] = h, \quad [h, p] = 0, \quad [h, q] = 0.$$

Then $\mathcal{D} = \mathfrak{z} \oplus \mathfrak{y}$ with $\mathfrak{z} = \mathrm{span}\{h\} = Z(\mathcal{D})$

$$\mathfrak{y} = \mathrm{span}\{p, q\} \quad \text{with} \quad [p, q]_0 = 0$$

and

$$\{0\} \to \mathfrak{z} \to \mathcal{D} \to \mathfrak{y} \to \{0\}$$

A favorite $\mathcal{D}$-module is $V = \mathbb{C}[x]$ with

p acting by $\dfrac{d}{dx}$

q acting by multiplication by x

h acting by multiplication by 1.

Another favorite $\mathcal{D}$-module is

$$V = \mathrm{span}\{b, x, \lambda_0 \text{[illegible]}\} \quad \text{with}$$

$$p = \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}, \quad q = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix} \quad h = \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$$

The Weyl algebra

$$\mathfrak{D} = \text{span}\{\underbrace{p_1, \ldots, p_g}_{\substack{\text{momentum}\\ \text{operators}}}, \underbrace{q_1, \ldots, q_g}_{\substack{\text{position}\\ \text{operators}}}, \hbar\}$$

with

$$[p_i, p_j] = 0, \qquad [q_i, q_j] = 0, \qquad \hbar \in Z(\mathfrak{D})$$

$$[p_i, q_j] = \delta_{ij}\hbar.$$

Then
$$\mathfrak{D} = \mathfrak{c} \oplus \mathfrak{g} \quad \text{with} \quad \mathfrak{c} = \text{span}\{\hbar\} = Z(\mathfrak{D}),$$

$\mathfrak{g} = \text{span}\{p_1, \ldots, p_g, q_1, \ldots, q_g\}$ with $[x_1, x_2]_0 = 0$ for $x_1, x_2 \in \mathfrak{g}$.

Weyl algebra modules are <u>$\mathfrak{D}$-modules</u>.

A favourite $\mathfrak{D}$-module is $V = \mathbb{C}[x_1, x_2, \ldots, x_g]$ with

p_i acting by $a\frac{\partial}{\partial x_i}$,

q_j acting by multiplication by x_j,

$\hbar$ acting by multiplication by a

(where $a \in \mathbb{C}^\times$ is fixed).

see Kac §9.3 and Gelfand-Manin Chapt 8 §1-2.

Virasoro algebra

$$Vir = \text{span}\{\dots, L_{-2}, L_{-1}, L_0, L_1, L_2, \dots\} \oplus \text{span}\{c\}$$

with

$$c \in Z(Vir), \qquad [L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n}\frac{1}{12}(n^3-n)c.$$

Then

$$\{0\} \longrightarrow \mathbb{C} \longrightarrow Vir \longrightarrow Witt \longrightarrow \{0\} \quad \text{with}$$

$\mathbb{C} = \text{span}\{c\}$, $\quad Witt = \text{span}\{\dots, L_{-2}, L_{-1}, L_0, L_1, L_2, \dots\}$

with

$$[L_m, L_n] = (m-n)L_{m+n}$$

and $f: Witt \times Witt \longrightarrow \mathbb{C}$ given by

$$f(L_m, L_n) = \delta_{m,-n}\frac{1}{12}(n^3-n)c.$$

A favorite Vir module is $V = \mathbb{C}[t, t^{-1}]$ (or $V = \mathbb{C}((t))$ or $V = C^\infty(S^1)$) with

$$L_m = -t^{m+1}\frac{d}{dt} \quad \text{and} \quad c = 0.$$

(following Kac §7.3).

Affine Lie algebras

$\mathring{\mathfrak{g}}$ a finite dimensional complex Lie algebra with bracket $[\,,\,]_0 : \mathring{\mathfrak{g}} \times \mathring{\mathfrak{g}} \to \mathring{\mathfrak{g}}$ and $\langle\,,\,\rangle_0 : \mathring{\mathfrak{g}} \times \mathring{\mathfrak{g}} \to \mathbb{C}$ a nondegenerate ad-invariant symmetric bilinear form.

$\mathring{\mathfrak{g}}[t,t^{-1}] = \mathbb{C}[t,t^{-1}] \otimes \mathring{\mathfrak{g}}$ is the <u>Loop Lie algebra</u>

$\mathring{\mathfrak{g}}[t,t^{-1}] = \mathrm{span}\{ x t^m \mid x \in \mathring{\mathfrak{g}},\ m \in \mathbb{Z} \}$ with

with $[xt^m, yt^n] = [x,y]_0 t^{m+n}$.

The <u>affine Lie algebra</u> is

$$\mathfrak{g} = \mathring{\mathfrak{g}}[t,t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d \quad \text{with}$$

$K \in Z(\mathfrak{g})$, $\quad [d, xt^m] = \left(t\frac{d}{dt}\right)(xt^m) = m x t^m$,

$[xt^m, yt^n] = [x,y]_0 t^{m+n} + \delta_{m,-n}\, m \langle x,y\rangle_0 K$.

Metaplectic groups

$\mathring{G}$ a reductive algebraic group.

$\mathring{G}(\mathbb{C}((t)))$ is the loop group

$\mathring{G}(\mathbb{Q}_p)$ is the p-adic group

Let $\mathbb{F} = \mathbb{C}((t))$ or $\mathbb{F} = \mathbb{Q}_p$.

A metaplectic group G is a central extension

$$\{1\} \to \mu_n \to G \to \mathring{G}(\mathbb{F}) \to \{1\}$$

where $\mu_n = \{n\text{th roots of unity}\}$
$= \{ e^{2\pi i j/n} \mid j = 0, 1, \ldots, n-1 \}$.

More generally a metaplectic group G is a central extension

$$\{1\} \to C \to G \to \mathring{G}(\mathbb{F}) \to \{1\}$$

where C is a quotient of

$$K_2(\mathbb{F}) = \frac{\mathbb{F}^\times \times \mathbb{F}^\times}{\langle u \otimes v \mid u + v = 1 \rangle}.$$

References

P. McNamara, Whittaker functions on Metaplectic groups, PhD Thesis MIT 2010.

J.-L. Brylinski and P. Deligne, Central extensions of reductive groups by K_2, Publ. Math. IHES 94 (2001), 5-85.

P. Deligne, Extension centrales de groupes algébriques simplement connexes et cohomologie galoisienne, Publ. Math. IHES 84 (1996) 35-89.

P. Deligne, Le symbole modéré, Publ. Math IHES 73 (1991) 147-181.

R. Steinberg, Lectures on Chevalley groups, Yale University 1967.

R. Steinberg, Générateurs, relations et revêtements de groupes algébriques, Colloq. Théorie des Groupes Algébriques (Bruxelles 1962) p. 113-127. Librairie Universitaire, Louvain 1962

C. Weibel, An introduction to homological algebra, Cambridge Univ. Press, 1994.

S. Gelfand and Yu. Manin, Homological algebra, Springer-Verlag 1999.