Langlands: The fundamental diagram

$$\begin{array}{ccccc} \mathbb{C}[\mathfrak{h}_{\mathbb{Z}}]^{W_0} = Z(H) & \xrightarrow{\sim} & \mathbf{1}_0 H \mathbf{1}_0 & \xrightarrow{\sim} & \varepsilon_0 H \mathbf{1}_0 \\ f & \longmapsto & f\mathbf{1}_0 & \longmapsto & A_\rho f \mathbf{1}_0 \end{array}$$

$$P_\lambda(0,t) \longleftarrow M_\lambda = \mathbf{1}_0 Y^\lambda \mathbf{1}_0$$

$$s_\lambda \longleftarrow C_\lambda \longleftarrow A_{\lambda+\rho} = \varepsilon_0 Y^{\lambda+\rho} \mathbf{1}_0$$

The players

H is the affine Hecke algebra

Generators: Y^λ for $\lambda \in \mathfrak{h}_{\mathbb{Z}}$ and T_w, for $w \in W_0$

Relations: $Y^\lambda Y^\mu = Y^\mu Y^\lambda = Y^{\lambda+\mu}$,

$$T_{s_i}^2 = (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) T_{s_i} + 1, \qquad \underbrace{T_{s_i} T_{s_j} T_{s_i} \cdots}_{m_{ij} \text{ factors}} = \underbrace{T_{s_j} T_{s_i} T_{s_j} \cdots}_{m_{ij} \text{ factors}}$$

$$T_{s_i} Y^\lambda = Y^{s_i\lambda} T_{s_i} + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \frac{Y^\lambda - Y^{s_i\lambda}}{1 - Y^{-\alpha_i^\vee}}$$

where $m_{ij} = \ldots$, s_i is $\ldots$, and $\alpha_i^\vee$ is $\ldots$.

H has basis

$$\{ Y^\lambda T_w \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}, w \in W_0 \}$$

so that, as vector spaces,

$$H = H_0 \otimes \mathbb{C}[\mathfrak{h}_{\mathbb{Z}}], \quad \text{where}$$

②

$H_0 = \mathrm{span}\{T_w \mid w \in W_0\}$ and

$\mathbb{C}[\mathfrak{h}_{\mathbb{Z}}] = \mathrm{span}\{y^\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}\}$ with $y^\lambda y^\mu = y^\mu y^\lambda = y^{\lambda+\mu}$.

$\mathbb{C}[\mathfrak{h}_{\mathbb{Z}}]^{W_0} =$ <u>symmetric</u> <u>functions</u>

$= \{f \in \mathbb{C}[\mathfrak{h}_{\mathbb{Z}}] \mid wf = f\}$ where $wy^\lambda = y^{w\lambda}$.

$Z(H) =$ <u>centre of H</u>

$= \{f \in H \mid$ if $h \in H$ then $hf = fh\}$

$\mathbb{1}_0 =$ projector onto the trivial representation

= the element of H_0 such that

$T_{s_i}\mathbb{1}_0 = t^{\frac{1}{2}}\mathbb{1}_0$ for $i = 1, 2, \ldots, n$

$\varepsilon_0 =$ projector onto the sign representation

= the element of H_0 such that

$T_{s_i}\varepsilon_0 = -t^{-\frac{1}{2}}\varepsilon_0$ for $i = 1, 2, \ldots n$

$W_0 =$ <u>the Weyl group</u>

Generators: $s_1, s_2, \ldots, s_n$

Relations: $s_i^2 = 1$, $\underbrace{s_i s_j s_i \cdots}_{m_{ij} \text{ factors}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij} \text{ factors}}$

<u>Initial Data</u>: W_0 acts on $\mathfrak{h}_{\mathbb{Z}}$

The Language

$\mathbb{C}[\mathfrak{h}_{\mathbb{Z}}]^{W_0} = Z(H)$ is "Bernstein's Theorem"

$\mathbb{C}[\mathfrak{h}_{\mathbb{Z}}]^{W_0} \cong \mathbf{1}_0 H \mathbf{1}_0$ is the "Satake isomorphism"

$\mathbf{1}_0 H \mathbf{1}_0 \cong \varepsilon_0 H \mathbf{1}_0$ is the "Whittaker isomorphism"
or the "Boson-Fermion correspondence"

$\varepsilon_0 H \mathbf{1}_0$ is the "space of Whittaker functions"
or the "Fock space"

s_λ is the "Weyl character" or "Schur function"

$P_\lambda(0,t)$ is the "Macdonald spherical function"
or "Hall-Littlewood polynomial".

$s_\lambda \longmapsfrom A_{\lambda+\rho}$ is the "Casselman-Shalika" formula.

Representation Theory

Let G be the reductive group with

Weyl group W_0 and coweight lattice $\mathfrak{h}_{\mathbb{Z}}$

Let $G^\vee$ be the "Langlands dual group of G",

i.e. the reductive group with

Weyl group W_0 and weight lattice $\mathfrak{h}_{\mathbb{Z}}$.

Let $L(\lambda)$ be the finite dimensional irreducible

$G^\vee(\mathbb{C})$-module with highest weight λ.

Then

$$\mathbb{C}[\mathfrak{h}_{\mathbb{Z}}]^{W_0} = Rep(G^\vee(\mathbb{C})),$$

the representation ring of finite dimensional

$G^\vee(\mathbb{C})$ representations,

and

$$s_\lambda = Res^{G^\vee}_{T^\vee}(L(\lambda))$$

where $T^\vee$ is the maximal torus of $G^\vee$

Geometry

$$\begin{array}{ccc} \mathbb{F} = \mathbb{Q}_p & & \mathbb{F} = \mathbb{C}((t)) \\ \cup\!\!\mid & \text{or} & \cup\!\!\mid \\ \mathfrak{o} = \mathbb{Z}_p \twoheadrightarrow k = \mathbb{F}_p & & \mathfrak{o} = \mathbb{C}[[t]] \twoheadrightarrow k = \mathbb{C} \end{array}$$

giving

$$\begin{array}{ccc} G = G(\mathbb{F}) & & \\ \cup\!\!\mid & & \\ K = G(\mathfrak{o}) & \xrightarrow{\Phi} & \mathring{G} = G(k) \\ \cup\!\!\mid & & \cup\!\!\mid \\ I = \Phi^{-1}(\mathring{B}) & \longrightarrow & \mathring{B} = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \in \mathring{G} \right\} \end{array}$$

If $\mathbb{F} = \mathbb{Q}_p$ then G is the <u>p-adic group</u>

If $\mathbb{F} = \mathbb{C}((t))$ then G is the <u>loop group</u>

G/I is the <u>affine flag variety</u>

G/K is the <u>loop Grassmannian</u>

I is a "minimal parahoric" or "Iwahori subgroup"

K is a "maximal parahoric".

<u>Then</u>

$$H \simeq C(I\backslash G/I) \simeq \mathrm{Perv}_I(G/I) \quad \text{and}$$

$$\mathbb{1}_0 H \mathbb{1}_0 = C(K\backslash G/K) \simeq \mathrm{Perv}_K(G/K),$$

where

$$C(I\backslash G/I) = \left\{ f: G \to \mathbb{C} \;\middle|\; \begin{array}{l} f(h_1 g h_2) = f(g) \\ \text{for } g \in G, \text{ and } h_1, h_2 \in I \end{array} \right\}$$

$$C(K\backslash G/K) = \left\{ f: G \to \mathbb{C} \;\middle|\; \begin{array}{l} f(k_1 g k_2) = f(g) \\ \text{for } g \in G \text{ and } k_1, k_2 \in K \end{array} \right\}$$

$\mathrm{Perv}_I(G/I)$ = I-equivariant perverse sheaves on G/I

$\mathrm{Perv}_K(G/K)$ = K-equivariant perverse sheaves on G/K.