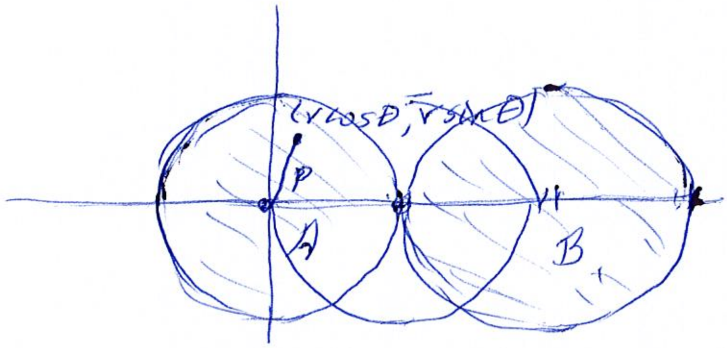
(1) Let A and B be the subsets of $\mathbb{R}^2$ given by

$$A = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\} \quad \text{and}$$

$$B = \{(x,y) \in \mathbb{R}^2 \mid (x-2)^2+y^2 < 1\}.$$

Determine whether $X = A \cup B$, $Y = \bar{A} \cup \bar{B}$, $Z = \bar{A} \cup B$ are connected.

The picture is



with

$$\bar{A} = \{(x,y) \mid x^2+y^2 \leq 1\} \quad \text{and}$$

$$\bar{B} = \{(x,y) \mid (x-2)^2+y^2 \leq 1\}.$$

The sets A, B, $\bar{A}$ and $\bar{B}$ are all path connected since

$$\bar{A} = \{(r\cos\theta, r\sin\theta) \mid r \in [0,1] \text{ and } \theta \in [0,2\pi)\}$$

and $p : [0,1] \to \bar{A}$ given by $p(t) = (rt\cos\theta, rt\sin\theta)$ is a path from $(0,0)$ to $(r\cos\theta, r\sin\theta)$.

$\bar{B}$ is a translate of $\bar{A}$ by 2 so it is also path connected.

Namely,

$$\overline{B} = (2,0) + \overline{A} = \{(r\cos\theta + 2, r\sin\theta) \mid r \in [0,1], \theta \in [0,2\pi]\}.$$

and $p' : [0,1] \to \overline{B}$ given by $p(t) = (rt\cos\theta + 2, rt\sin\theta)$ is a path from $(2,0)$ to $(r\cos\theta + 2, r\sin\theta)$.

(a) Let $X = A \cup B$. Then

$$X = A \cup B, \quad A \cap B = \phi, \quad A \neq \phi \text{ and } B \neq \phi$$

and so X is not connected.

(b) Let $Y = \overline{A} \cup \overline{B}$. Then $\overline{A}$ is path connected and $\overline{B}$ is path connected and $(1,0) \in \overline{A} \cap \overline{B}$. It follows that $Y = \overline{A} \cup \overline{B}$ is path connected and connected.

(c) Let $Z = \overline{A} \cup B = \overline{A} \cup ((1,0) \cup B)$

Then every point of $\overline{A}$ is path connected to $(1,0)$
Every point of $(1,0) \cup B$ is path connected to $(1,0)$.

So Z is path connected and connected (since path connected implies connected, see the Rubinstein notes, the sentence between Definition 8.13 and Example 8.14).

(2) Let X and Y be topological spaces and assume Y is Hausdorff. Let
$f: X \to Y$ and $g: X \to Y$ be continuous functions.

(a) Show that $\{x \in X \mid f(x) = g(x)\}$ is closed in X.

To show: $\{x \in X \mid f(x) = g(x)\}^c$ is open.

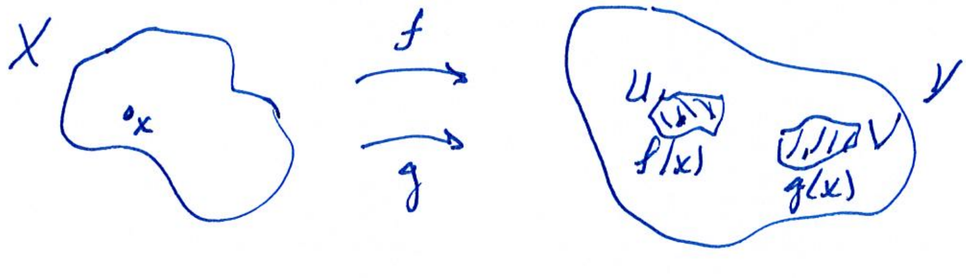
To show: $\{x \in X \mid f(x) \neq g(x)\}$ is open.

Let $A = \{x \in X \mid f(x) \neq g(x)\}$. Let $x \in A$.

To show: x is an interior point of A.

Since Y is Hausdorff and $f(x) \neq g(x)$ there exist open sets U and V of Y with

$f(x) \in U$, $g(x) \in V$ and $U \cap V = \phi$.



To show: there exists an open set B of X such that $x \in B$ and $B \subseteq A$.

Let

$N = f^{-1}(U)$ and $P = g^{-1}(V)$ and let $B = N \cap P$.

To show: (aa) B is open

(ab) $x \in B$

(ac) $B \subseteq A$.

(aa) Since f is continuous, $N = f^{-1}(U)$ is open.

Since g is continuous, $P = g^{-1}(V)$ is open.

So $B = N \cap P$ is open.

(ab) Since $f(x) \in U$, then $x \in f^{-1}(U) = N$.

Since $g(x) \in V$, then $x \in g^{-1}(V) = P$.

So $x \in N \cap P = B$.

(ac) To show: $B \subseteq A$.

To show: If $x' \in B$ then $x' \in A$.

Assume $x' \in B$.

To show: $f(x') = g(x')$.

Since $x' \in B$, then $f(x') \in f(N) \subseteq U$

and $g(x') \in g(P) \subseteq V$.

Since $U \cap V = \phi$ then $f(x') \neq g(x')$

So $x' \in A$.

So $B \subseteq A$.

So x is an interior point of A.

So A is open and $\{x \in X \mid f(x) = g(x)\}$ is closed.

(b) Assume $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are continuous.

To show: $f-g$ is continuous.

The function $f-g$ is the composition of

$$\begin{matrix} X \xrightarrow{f \times g} \mathbb{R}\times\mathbb{R} & & \mathbb{R}\times\mathbb{R} \to \mathbb{R} \\ x \mapsto (f(x), g(x)) & \text{and} & (x,y) \mapsto x-y \end{matrix}$$

and the composition of continuous functions is continuous.

To show: (ba) $\begin{matrix} f \times g : X \to \mathbb{R}\times\mathbb{R} \\ x \mapsto (f(x), g(x)) \end{matrix}$ is continuous.

(bb) $\begin{matrix} \mathbb{R}\times\mathbb{R} \to \mathbb{R} \\ (x,y) \mapsto x-y \end{matrix}$ is continuous.

(ba) This is part 1 of theorem 3.6 in the Rubinstein notes. No proof is given there, so it might be desirable to provide a proof.

(bb) To show: If $(x_1, y_1), (x_2, y_2), \ldots$ is a sequence in $\mathbb{R}^2$ and $\lim_{n\to\infty} (x_n, y_n) = (x,y)$ then

$$\lim_{n\to\infty} (x_n - y_n) = x - y.$$

To show: $\lim_{n\to\infty} (x_n - y_n) = \left(\lim_{n\to\infty} x_n\right) - \left(\lim_{n\to\infty} y_n\right).$

This is a fact usually proved in 2nd year Real analysis. Again, it might be desirable to provide a proof.

(c) Show that if $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ are continuous then

$$\{x \in X \mid f(x) < g(x)\} \text{ is open.}$$

<u>Proof</u> Let $B = \{x \in X \mid f(x) < g(x)\}$.

Then $B = \{x \in X \mid (f-g)(x) < 0\} = (f-g)^{-1}(\mathbb{R}_{<0})$.

Since $\mathbb{R}_{<0}$ is open in $\mathbb{R}$ and, by (b), $f-g$ is continuous,

$B = (f-g)^{-1}(\mathbb{R}_{<0})$ is open. //

(3) Let X be a Banach space. For $a \in X$ and $r \in \mathbb{R}_{>0}$ let
$$S(a,r) = \{x \in X \mid d(x,a) = |x-a| = r\}.$$

(a) Show that $S(a,r)$ is nowhere dense.

Let $a \in X$ and $r \in \mathbb{R}_{>0}$.

To show: $(\overline{S(a,r)})^\circ = \phi$

To show: (aa) $\overline{S(a,r)} = S(a,r)$

(ab) $S(a,r)^\circ = \phi$.

(aa) To show: $S(a,r)$ is closed.

$$S(a,r) = f^{-1}(\{r\}), \text{ where } \begin{array}{rl} f: X & \to \mathbb{R} \\ x & \mapsto d(x,a) \end{array}$$

Since f is continuous (see Theorem 3.6 of the Rubinstein notes) and $\{r\}$ is closed, then
$$S(a,r) = f^{-1}(\{r\}) \text{ is closed.}$$

(ab) To show: $S(a,r)^\circ = \phi$.

To show: If $x \in S(a,r)$ then x is not an interior point of $S(a,r)$.

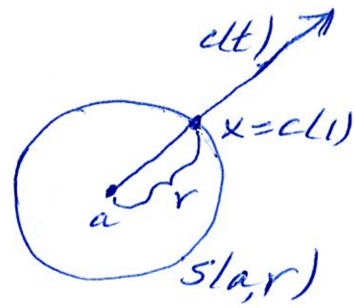
Assume $x \in S(a,r)$.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then $B(x,\varepsilon) \not\subseteq S(a,r)$.

Assume $\varepsilon \in \mathbb{R}_{>0}$.

3. For $t \in \mathbb{R}_{\geq 0}$ let

$$c(t) = a + t(x-a)$$



Since

$$d(c(t), x) = \|a + t(x-a) - x\| = \|(t-1)(x-a)\|$$
$$= |t-1|\,\|x-a\| = |t-1| \cdot r,$$

then $c(t) \in B(x, \varepsilon)$ for $r \cdot |t-1| < \varepsilon$.

So $c(t) \in B(x,\varepsilon)$ for $1 - \frac{\varepsilon}{r} < t < 1 + \frac{\varepsilon}{2}$.

Since

$$d(c(t), a) = \|a + t(x-a) - a\| = t\|x-a\| = t \cdot r,$$

then $c(t) \in S(a,r)$ only if $t = 1$.

So $B(x,\varepsilon) \not\subseteq S(a,r)$.

So x is not an interior point of $S(a,r)$.

So $S(a,r)^\circ = \phi$.

So $(\overline{S(a,r)})^\circ = S(a,r)^\circ = \phi$ and

$S(a,r)$ is nowhere dense in X.

(b) Show that $X \neq \bigcup_{n \in \mathbb{Z}_{>0}} S_n$, where S_n are spheres.

To show: If $a_1, a_2, \ldots \in X$ and $r_1, r_2, \ldots \in \mathbb{R}_{>0}$ then $X \neq \bigcup_{n \in \mathbb{Z}_{>0}} S(a_n, r_n)$.

Assume $a_1, a_2, \ldots \in X$ and $r_1, r_2, \ldots \in \mathbb{R}_{>0}$.

By part (a), each $S(a_n, r_n)$ is nowhere dense in X.

By the Baire theorem,

X is not a countable union of nowhere dense sets.

So, ~~by (a)~~ $X \neq \bigcup_{n \in \mathbb{Z}_{>0}} S(a_n, r_n)$.

(c) Give a geometric interpretation of the result in (b) when $X = \mathbb{R}^2$ with the Euclidean norm.

If $X = \mathbb{R}^2$ then $S(a,r)$ is a circle with centre a and radius r.

So (b) says that the plane $\mathbb{R}^2$ cannot be completely covered by a sequence of circles.

(d) Show that the result of (b) does not hold in every complete metric space X.

To show: There exists a completes metric space X and spheres $S(a_1, r_1), S(a_2, r_2) \ldots$ in X with ~~X~~ $X = \bigcup_{n \in \mathbb{Z}_{>0}} S(a_n, r_n)$

Let $X = \mathbb{Z}$ (so X is an infinite set with the discrete topology).

Then $\mathbb{Z}$ is a complete metric space (since it is a closed subset of the complete metric space $\mathbb{R}$).

Let $0 = a_1 = a_2 = \cdots$ and $r_n = n$ for $n \in \mathbb{Z}_{>0}$.

Then $S(a_n, r_n) = S(0, n) = \{-n, n\}$ in $\mathbb{Z}$.

So $\mathbb{Z} = \bigcup_{n \in \mathbb{Z}_{>0}} \{-n, n\} = \bigcup_{n \in \mathbb{Z}_{>0}} S(0, n)$.

(4) Prove that if X and Y are path connected then $X \times Y$ is also path connected.

<u>Proof</u> Assume X and Y are path connected.

To show: $X \times Y$ is path connected

To show: If $(x_1, y_1), (x_2, y_2) \in X \times Y$ then there exists a path $p : [0,1] \to X \times Y$ connecting (x_1, y_1) and (x_2, y_2).

Assume $(x_1, y_1) \in X \times Y$ and $(x_2, y_2) \in X \times Y$.

Since X and Y are path connected we know that there exist continuous functions

$$p_1 : [0,1] \to X \quad \text{and} \quad p_2 : [0,1] \to Y$$

with $\begin{matrix} p_1(0) = x_1 \\ p_1(1) = x_2 \end{matrix}$ and $\begin{matrix} p_2(0) = y_1 \\ p_2(1) = y_2. \end{matrix}$

To show: There exists a continuous function $p : [0,1] \to X \times Y$ with $p(0) = (x_1, y_1)$ and $p(1) = (x_2, y_2)$.

Let

$p : [0,1] \to X \times Y$ be given by $p(t) = (p_1(t), p_2(t))$.

Then $p(0) = (p_1(0), p_2(0)) = (x_1, y_1)$,

$p(1) = (p_1(1), p_2(1)) = (x_2, y_2)$, and

p is continuous by Theorem 3.6 in the Rubinstein notes.

More specifically Theorem 3.6 part (a) states:

"Let f be a function from X to Y_1 and let g be a function from X to Y_2. Define the function h from X to the product $Y_1 \times Y_2$ by

$$h(x) = (f(x), g(x)), \quad \text{for } x \in X.$$

Then h is continuous if and only if both functions f and g are continuous"

This statement is not proved in the Rubinstein notes so it might be desirable to provide a proof.