(1) Let A and B be bonded subsets of a metric space (X,d) such that $A\cap B=\phi$. Show that

$$\mathrm{diam}(A\cup B)\le \mathrm{diam}(A)+\mathrm{diam}(B)$$

What can you say if A and B are disjoint?

<u>Solution</u> The definition of $\mathrm{diam}(A)$ is

$$\mathrm{diam}(A)=\sup\{d(x,y)\mid x,y\in A\}$$

Assume $A\subseteq X$ and $B\subseteq X$ and $A\cap B=\phi$ and

$$\mathrm{diam}(A)<\infty \text{ and } \mathrm{diam}(B)<\infty.$$

To show: $\mathrm{diam}(A\cup B)\le \mathrm{diam}(A)+\mathrm{diam}(B)$.

To show: $\mathrm{diam}(A)+\mathrm{diam}(B)$ is an upper bound of $\{d(x,y)\mid x,y\in A\cup B\}$.

To show: If $x,y\in A\cup B$ then $d(x,y)\le \mathrm{diam}(A)+\mathrm{diam}$.

Assume $x,y\in A\cup B$

<u>Case 1</u>: $x,y\in A$. Then

$$d(x,y)\le \mathrm{diam}(A)\le \mathrm{diam}(A)+\mathrm{diam}(B)$$

<u>Case 2</u>: $x,y\in B$. Then

$$d(x,y)\le \mathrm{diam}(B)\le \mathrm{diam}(A)+\mathrm{diam}(B).$$

<u>Case 3</u>: $x \in A$ and $y \in B$. Let $z \in A \cap B$. Then

$$d(x,y) \leq d(x,z) + d(z,y) \leq \operatorname{diam}(A) + \operatorname{diam}(B).$$

<u>Case 4</u>: $x \in B$ and $y \in A$. Let $z \in A \cap B$. Then

$$d(x,y) \leq d(x,z) + d(z,y) \leq \operatorname{diam}(A) + \operatorname{diam}(B).$$

So $\operatorname{diam}(A) + \operatorname{diam}(B)$ is an upper bound of
$\{(x,y) \mid x, y \in A \cup B\}$.

So $\operatorname{diam}(A \cup B) \leq \operatorname{diam}(A) + \operatorname{diam}(B)$.

(2) Let $X = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$

Let $d_\infty : X \times X \to \mathbb{R}_{\geq 0}$ and $d_1 : X \times X \to \mathbb{R}_{\geq 0}$ be given by

$$d_\infty(f,g) = \sup\{|f(x) - g(x)| \mid x \in [0,1]\} \quad \text{and}$$

$$d_1(f,g) = \int_0^1 |f(x) - g(x)|\, dx.$$

Let $f_1, f_2, \ldots \in X$ be given by

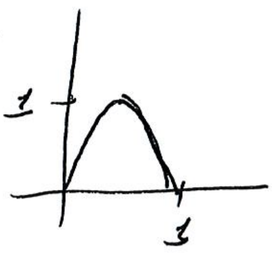
$$f_n : [0,1] \to \mathbb{R} \quad \text{given by } f_n(x) = n x^n (1-x).$$

(a) Determine whether $\{f_n\}$ converges in (X, d_1).

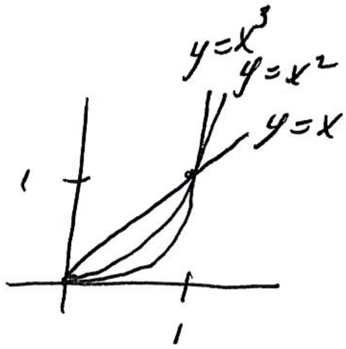
(b) Determine whether $\{f_n\}$ converges in (X, d_∞).

The graph of

$y = x - x^2$

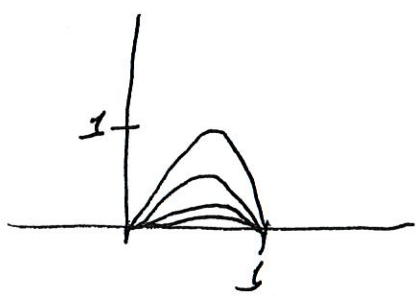


and $y = x^{n-1}$



help to determine the graph of

$y = n x^n (1-x) = n x^{n-1}(x - x^2)$



Since $\lim_{n\to\infty} n x^{n-1} = 0$ for $x \in [0,1)$ and $n x^{n-1} = n$ for $x = 1$

and $x - x^2 = 0$ for $x = 1$, then $\lim_{n\to\infty} n x^{n-1}(x - x^2) = 0$ for $x \in [0,1]$.

So the pointwise limit of $\{f_n\}$ is the zero function

$f:[0,1]\to\mathbb{R}$ given by $f(x)=0$.

$\{f_n\}$ converges in (X,d_1) if $\lim\limits_{n\to\infty} d_1(f_n,f)=0$.

$\{f_n\}$ converges in (X,d_∞) if $\lim\limits_{n\to\infty} d_\infty(f_n,f)=0$.

<u>Compute</u>: $d_1(f_n,f)$.

$$d_1(f_n,f)=\int_0^1|f_n(x)-f(x)|\,dx=\int_0^1(nx^n(1-x)-0)\,dx$$

$$=\int_0^1(nx^n-nx^{n+1})\,dx=\left(n\frac{x^{n+1}}{n+1}-n\frac{x^{n+2}}{n+2}\right)\Bigg|_{x=0}^{x=1}$$

$$=\left(\frac{n}{n+1}-\frac{n}{n+2}\right)=\frac{n}{(n+1)(n+2)}.$$

So $\lim\limits_{n\to\infty} d_1(f_n,f)=\lim\limits_{n\to\infty}\frac{n}{(n+1)(n+2)}=\lim\limits_{n\to\infty}\frac{1}{(1+\frac{1}{n})}\,\frac{1}{(n+2)}$

$$=1\cdot 0=0.$$

So $\{f_n\}$ converges in (X,d_1).

<u>Compute</u> <u>$d_\infty(f_n,f)$</u>:

To compute $d_\infty(f_n,f)=\sup\{|nx^n(1-x)-0| \mid x\in[0,1]\}$

find the maximum of

$f_n(x)=nx^n(1-x)$ on the interval $[0,1]$.

This maximum occurs at $x=0$ or $x=1$ or at a critical point. Since

$$\frac{df_n}{dx} = \frac{d\, nx^n(1-x)}{dx} = n^2x^{n-1} - n(n+1)x^n = nx^{n-1}(n-nx-x)$$

the critical points are at $x=0$ and $x=\frac{n}{n+1}$.

Since $f_n(0)=0$ and $f_n(1)=0$ and $f_n\left(\frac{n}{n+1}\right) = n\left(\frac{n}{n+1}\right)^n\left(1-\frac{n}{n+1}\right)$

the maximum of f_n is $\left(\frac{n}{n+1}\right)^{n+1} = \left(1-\frac{1}{n+1}\right)^{n+1}$.

So
$$\begin{aligned} d_\infty(f_n, f) &= \sup\{|f_n(x)-f(x)| \mid x\in[0,1]\} \\ &= \sup\{nx^n(1-x) \mid x\in[0,1]\} \\ &= \left(1-\frac{1}{n+1}\right)^{n+1}. \end{aligned}$$

and
$$\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^{n+1} = \lim_{n\to\infty} e^{\log\left(\left(1-\frac{1}{n+1}\right)^{n+1}\right)}$$

$$= \lim_{n\to\infty} e^{(n+1)\log\left(1-\frac{1}{n+1}\right)} = \lim_{n\to\infty} e^{-(n+1)\left(\frac{1}{n+1}+\frac{1}{2}\left(\frac{1}{n+1}\right)^2+\frac{1}{3}\left(\frac{1}{n+1}\right)^3+\cdots\right)}$$

since $-\log(1-x) = \int\frac{1}{1-x}\,dx = \int(1+x+x^2+\cdots)\,dx = x+\frac{1}{2}x^2+\frac{1}{3}x^3+\cdots$

So $\lim_{n\to\infty} d_\infty(f_n,f) = \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^{n+1} = e^{-(1+\frac{1}{2}\cdot 0+\frac{1}{3}\cdot 0^2+\cdots)} = e^{-1}$.

So $\{f_n\}$ does <u>not</u> converge in (X, d_∞).

(3) Let X and Y be topological spaces.
Let $A \subseteq X$ and $B \subseteq Y$. Show that $\overline{A} \times \overline{B} = \overline{A \times B}$.

Proof To show: (a) $\overline{A} \times \overline{B} \subseteq \overline{A \times B}$

(b) $\overline{A \times B} \subseteq \overline{A} \times \overline{B}$.

(a) Assume $(x,y) \in \overline{A} \times \overline{B}$.

To show: $(x,y) \in \overline{A \times B}$.

To show: (x,y) is a close point of $A \times B$.

Let N be a neighborhood of (x,y) in $X \times Y$.

By the definition of the product topology on $X \times Y$
there exist N_x, a neighborhood of x in X,
and N_y, a neighborhood of y in Y,
such that $N_x \times N_y \subseteq N$.

Since $x \in \overline{A}$ there exists $a \in A$ with $a \in N_x$.

Since $y \in \overline{B}$ there exists $b \in B$ with $b \in N_y$.

So $(a,b) \in N_x \times N_y \subseteq N$ and $(a,b) \in A \times B$.

So (x,y) is a close point of $A \times B$.

So $(x,y) \in \overline{A \times B}$

So $\overline{A} \times \overline{B} \subseteq \overline{A \times B}$.

(b) To show: $\overline{A \times B} \subseteq \overline{A} \times \overline{B}$.

Assume $(x, y) \in \overline{A \times B}$.

To show: $(x, y) \in \overline{A} \times \overline{B}$.

To show: $x \in \overline{A}$ and $y \in \overline{B}$.

Let N_x be a neighborhood of $x \in X$ and let N_y be a neighborhood of $y \in Y$.

Then $N_x \times N_y$ is a neighborhood of $(x, y) \in X \times Y$.

Since (x, y) is a close point of $A \times B$, there exists $(a, b) \in A \times B$ with $(a, b) \in N_x \times N_y$

So $a \in N_x$ and $b \in N_y$ and $a \in A$ and $b \in B$.

So x is a close point of A and y is a close point of B.

So $x \in \overline{A}$ and $y \in \overline{B}$.

So $(x, y) \in \overline{A} \times \overline{B}$.

(4) Let (X,d) be a metric space and let $A \subseteq X$ with $A \neq \phi$. For $x \in X$ let
$$d(x,A) = \inf\{d(x,a) \mid a \in A\}.$$

(a) Prove that $\bar{A} = \{x \in X \mid d(x,A) = 0\}$

To show: (aa) $\{x \in X \mid d(x,A) = 0\} \subseteq \bar{A}$

(ab) $\bar{A} \subseteq \{x \in X \mid d(x,A) = 0\}$.

(aa) Assume $x \in X$ and $d(x,A) = 0$.

To show: $x \in \bar{A}$.

Let N be a neighborhood of x in X.

Then there exists $\varepsilon \in \mathbb{R}_{>0}$ such that $B(x,\varepsilon) \subseteq N$

~~Let $a \in A$~~ Since $d(x,A) = \inf\{d(x,a) \mid a \in A\} = 0$,

there exists $a \in A$ such that $d(x,a) < \varepsilon$.

Then $a \in B(x,\varepsilon) \subseteq N$ and $a \in A$.

So x is a close point of A.

So $\{x \in X \mid d(x,A) = 0\} \subseteq \bar{A}$.

(ab) To show: $\bar{A} \subseteq \{x \in X \mid d(x,A) = 0\}$.

Let $x \in \bar{A}$.

So x is a close point of A.

To show: $d(x,A) = 0$.

Let $\varepsilon \in \mathbb{R}_{>0}$

Then $B(x,\varepsilon)$ is a neighborhood of x in X.

Since x is a close point of A

there exists $a \in A$ such that $a \in B(x, \varepsilon)$.

So $d(x,a) < \varepsilon$.

So $d(x, A) < \varepsilon$ for all $\varepsilon \in \mathbb{R}_{>0}$.

So $d(x, A) = 0$.

So $x \in \{x \in X \mid d(x, A) = 0\}$.

So $\bar{A} \subseteq \{x \in X \mid d(x, A) = 0\}$.

Thus $\bar{A} = \{x \in X \mid d(x, A) = 0\}$.

(b) Show that if $x, y \in X$ then $|d(x, A) - d(y, A)| \leq d(x, y)$.

Assume $x, y \in X$.

To show: (ba) $d(x, A) - d(y, A) \leq d(x, y)$

(bb) $-(d(x, A) - d(y, A)) \leq d(x, y)$.

(ba) Since $d(x, A)$ is a lower bound of $\{d(x,a) \mid a \in A\}$,

if $a \in A$ then $d(x, A) \leq d(x, a)$.

Using $d(x,a) \leq d(x,y) + d(y,a)$,

if $a \in A$ then $d(x, A) \leq d(x,y) + d(y,a)$.

So $d(x, A)$ is a lower bound of $\{d(x,y) + d(y,a) \mid a \in A\}$.

Since $d(x,y) + d(y, A)$ is the <u>greatest</u> lower bound of

$\{d(x,y) + d(y,a) \mid a \in A\}$ then

$$d(x,A) \le d(x,y) + d(y,A).$$

So $d(x,A) - d(y,A) \le d(x,y)$.

So $d(y,A) - d(x,A) \le d(y,x) = d(x,y)$

So $-(d(x,A) - d(y,A)) \le d(x,y)$.

So $d(x,A) - d(y,A) \le d(x,y)$ and $-(d(x,A) - d(y,A)) \le d(x,y)$

So $|d(x,A) - d(y,A)| \le d(x,y)$.

(c) Let $f: X \to \mathbb{R}$ be given by $f(x) = d(x,A)$.

Show that f is continuous.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ ~~the~~ and $x \in X$ then there exists $\delta \in \mathbb{R}_{>0}$ such that

if $y \in X$ and $d(x,y) < \delta$ then $d(f(x), f(y)) < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$ and $x \in X$.

To show: There exists $\delta \in \mathbb{R}_{>0}$ such that

if $y \in X$ and $d(x,y) < \delta$ then $d(f(x), f(y)) < \varepsilon$.

Let $\delta = \varepsilon$.

To show: If $y \in X$ and $d(x,y) < \delta$ then $d(f(x), f(y)) < \varepsilon$.

Assume $y \in X$ and $d(x,y) < \delta$.

To show: $d(f(x), f(y)) < \varepsilon$.

By part (b),

$$d(f(x), f(y)) = |d(y,A) - d(x,A)| \le d(x,y) < \delta = \varepsilon.$$

So f is continuous

(d) Assume $x \notin \bar{A}$ and let $U = \{y \in X \mid d(y,A) < d(x,A)\}$.

Show that (da) $x \notin U$

(db) U is open

(dc) $\bar{A} \subseteq U$.

(da) Let $D = d(x,A)$.

Since $x \notin \bar{A}$ and, by part (a), $\bar{A} = \{y \in X \mid d(y,A) = 0\}$

then $d(x,A) \neq 0$.

So $D \neq 0$.

We know $U = \{y \in X \mid d(y,A) < D\}$.

(da) Since $d(x,A) = D$, $x \notin U$.

(db) Since $U = f^{-1}(\mathbb{R}_{<D}) = f^{-1}((-\infty, D))$

and f is continuous, then U is open.

(dc) By part (a),

$\bar{A} = \{y \in X \mid d(y,A) = 0\} \subseteq \{y \in X \mid d(y,A) < D\} = U$.

So $\bar{A} \subseteq U$.