§6.1 Fredholm Theory

Let H be a Hilbert space.

A linear operator $K : H \to H$ is compact if K satisfies:

if $u_1, u_2, \ldots$ is a sequence in H such that $\{u_1, u_2, \ldots\}$ is bounded

then there exists a subsequence $(u_{n_1}, u_{n_2}, \ldots)$ such that $(Ku_{n_1}, Ku_{n_2}, \ldots)$ converges.

Theorem 6.1 Let H be a Hilbert space over $\mathbb{R}$.

Let $K : H \to H$ be a compact linear operator.

(a) $\dim(\ker(1-K)) < \infty$.

(b) $\mathrm{im}\,(1-K)$ is closed.

(c) $\mathrm{im}\,(1-K) = \ker(1-K^*)^\perp$

(d) $\ker(1-K) = \{0\}$ if and only if $\mathrm{im}\,(1-K) = H$.

(e) $\dim(\ker(1-K)) = \dim(\ker(1-K^*))$.

The Fredholm alternative is

Case 1: $\ker(1-K) = 0$

Case 2 $\ker(1-K) \neq 0$.

§6.2 Spectra

Let H be a Hilbert space over $\mathbb{R}$ and let $\Lambda : H \to H$ be a bounded linear operator.

The resolvent set, spectrum, point spectrum and essential spectrum of Λ are defined by

$\rho(\Lambda) = \{\eta \in \mathbb{R} \mid \eta - \Lambda \text{ is bijective}\}$ resolvent set

$\sigma(\Lambda) = \{\eta \in \mathbb{R} \mid \eta - \Lambda \text{ is not bijective}\}$ spectrum

$\sigma_p(\Lambda) = \{\eta \in \mathbb{R} \mid \eta - \Lambda \text{ is not injective}\}$ point spectrum

$\sigma_e(\Lambda) = \left\{\eta \in \mathbb{R} \;\middle|\; \begin{array}{l} \eta - \Lambda \text{ is injective} \\ \text{and not surjective} \end{array}\right\}$ essential spectrum

HW Show that if $\eta \in \mathbb{R}$ and $\eta - \Lambda$ is bijective then $(\eta - \Lambda)^{-1}$ is continuous.

HW Show that $\eta \in \sigma_p(\Lambda)$ if and only if there exists $w \in H$ with $w \neq 0$ and $\Lambda w = \eta w$ i.e, there exists an eigenvector w of Λ with eigenvalue η.

<u>Theorem</u> <u>6.3</u> Let H be an infinite dimensional Hilbert space. Let $K: H \to H$ be a compact linear operator. Then

(a) $0 \in \sigma(K)$,

(b) $\sigma(K) = \sigma_p(K) \cup \{0\}$,

(c) If $\sigma_p(K)$ is not finite then

$$\sigma_p(K) = \{\lambda_k \mid k \in \mathbb{Z}_{>0}\} \text{ with } \lim_{k\to\infty} \lambda_k = 0.$$

<u>§6.3</u> <u>Self adjoint operators</u>

Let H be a Hilbert space over $\mathbb{R}$ and let $\Lambda: H \to H$ be a bounded linear operator. The operator $\Lambda: H \to H$ is <u>symmetric</u> if Λ satisfies

$$\text{if } x, y \in H \text{ then } \langle \Lambda x, y\rangle = \langle x, \Lambda y\rangle.$$

<u>Lemma 6.5</u> Let H be a Hilbert space over $\mathbb{R}$. Let $\Lambda: H \to H$ be a bounded linear self adjoint operator. Let

$$m = \inf\{\langle \Lambda u, u\rangle \mid u \in H \text{ and } \|u\| = 1\} \text{ and}$$
$$M = \sup\{\langle \Lambda u, u\rangle \mid u \in H \text{ and } \|u\| = 1\}.$$

Then

$$\sigma(\Lambda) \subseteq [m, M], \qquad m, M \in \sigma(\Lambda) \text{ and } \|\Lambda\| = \max\{-m, M\}.$$

Theorem 6.6 (Hilbert-Schmidt).
Let H be a Hilbert space over $\mathbb{R}$.
Assume H is separable.
Let $K: H \to H$ be a compact symmetric linear operator. Then

there exists a countable orthonormal basis B of H consisting of eigenvectors of K.

The construction of B:

Let $\{\eta_1, \eta_2, \ldots\}$ be the eigenvalues of K and

$H_0 = \ker(K)$, $H_1 = \ker(K - \eta_1)$, $H_2 = \ker(K - \eta_2)$, ...

Let B_k be an orthonormal basis of H_k

and let $B = \bigcup_{k \in \mathbb{Z}_{\geq 0}} B_k$.

Remark 6.7 Let H be a Hilbert space over $\mathbb{R}$.
Let $K: H \to H$ be a compact linear self adjoint operator. Assume H is separable.
Let $f \in H$. Show that if $1 \notin \sigma(K)$ then

$u - Ku = f$ has a unique solution

given by

$$u = \sum_{k \in \mathbb{Z}_{\geq 0}} \frac{\langle f, w_k \rangle}{1 - \lambda_k} w_k, \qquad \text{where}$$

$\{w_1, w_2, \dots\}$ is an orthonormal basis of H consisting of eigenvectors of K.

Examples of when $\overline{\mathrm{span}(S)} = X$

(1) Let E be a compact metric space. Let

$$C(E) = \{f : E \to \mathbb{R} \mid f \text{ is continuous}\}.$$

Let $S \subseteq C(E)$ such that

span(S) is an algebra that contains the constant function 1 and separates points.

Show that $\overline{\mathrm{span}(S)} = C(E)$.

(2) Let H be a separable Hilbert space.

Let $\Lambda : H \to H$ be a compact self adjoint operator.

Let $S \subseteq H$ such that

(a) span(S) contains all eigenvectors of Λ,

(b) span$(S) \supseteq \ker(\Lambda)$.

Show that $\overline{\mathrm{span}(S)} = H$.