# Examples

(1) $\mathbb{R}^n = \{\vec{x} = (x_1, x_2, \dots, x_N) \mid x_i \in \mathbb{R}\}$ with norm

$$\|\vec{x}\|_p = \left(|x_1|^p + |x_2|^p + \cdots + |x_N|^p\right)^{1/p} \quad \text{or}$$

$$\|\vec{x}\|_\infty = \sup\{|x_1|, |x_2|, \dots, |x_N|\}$$

(2) $\ell^p = \{\vec{x} = (x_1, x_2, \dots) \mid x_i \in \mathbb{R}, \sum_{k \in \mathbb{Z}_{>0}} |x_k|^p < \infty\}$

with norm

$$\|\vec{x}\|_p = \Big(\sum_{k \in \mathbb{Z}_{>0}} |x_k|^p\Big)^{1/p}.$$

(2b) $\ell^\infty = \{\vec{x} = (x_1, x_2, \dots) \mid x_i \in \mathbb{R}, \|\vec{x}\|_\infty < \infty\}$

with norm

$$\|\vec{x}\|_\infty = \sup\{|x_k| \mid k \in \mathbb{Z}_{>0}\}.$$

(3) Let $a, b \in \mathbb{R}$ with $a < b$ and let

$$\mathcal{C}^0([a,b]) = \{f : [a,b] \to \mathbb{R} \mid f \text{ is continuous}\}$$

with norm

$$\|f\|_{C^0} = \sup\{|f(x)| \mid x \in [a,b]\}.$$

(4) Let $\Omega$ be an open subset of $\mathbb{R}^n$ and $p \in \mathbb{R}_{\geq 1}$.
Let

$$L^p(\Omega) = \{f : \Omega \to \mathbb{R} \mid f \text{ is Lebesgue measurable and } \|f\|_p <$$

with norm $\|f\|_{L^p} = \left(\int_\Omega |f(x)|^p \, dx\right)^{1/p}$ where $f \sim \tilde{f}$ if $\mu(\{x \in \Omega \mid f(x) \neq \tilde{f}(x)\}) =$

(4b) Let $\Omega$ be an open set in $\mathbb{R}^n$. Let

$$L^\infty(\Omega) = \{ f : \Omega \to \mathbb{R} \mid f \text{ is measurable and } \|f\|_\infty < \infty \}$$

with norm

$$\|f\|_{L^\infty} = \operatorname{ess\,sup} \{ |f(x)| \mid x \in \Omega \}.$$

$$= \inf \{ \alpha \in \mathbb{R} \mid |f(x)| < \alpha \text{ for a.e. } x \in \Omega \}.$$

$$= \inf \{ \alpha \in \mathbb{R} \mid \mu(x \in \Omega \mid f(x) < \alpha) = \mu(\Omega) \}.$$

$$= \inf \{ \alpha \in \mathbb{R} \mid \mu(\{ x \in \Omega \mid f(x) \geq \alpha \}) = 0 \}.$$

<u>HW</u> <u>Example 2.10</u> Construct a sequence $(f_1, f_2, f_3, \dots)$ of polynomials $f_k : [0,1] \to \mathbb{R}$ such that

$$\lim_{n \to \infty} \|f\|_{L^1} = 0 \text{ and } \lim_{n \to \infty} \|f\|_{C^0} = 1.$$

<u>HW</u> <u>Example 2.10</u> Let $f_k : [0,1] \to \mathbb{R}$ be given by $f_k(x) = x^k$. Show that

(a) in the $L^1$ norm $(f_1, f_2, \dots)$ converges and find the limit.

(b) in the $C^0$ norm $(f_1, f_2, \dots)$ does not converge.

(c) Graph $f_1, f_2, f_3, \dots$

<u>HW</u> <u>Remark 2.8</u> Let $e_i = (0, 0, \dots 0, \overset{i\text{th}}{1}, 0, 0, \dots)$ for $i \in \mathbb{Z}_{>0}$. Show that

$$\operatorname{span}\{e_k \mid k \in \mathbb{Z}_{>0}\} \subsetneq \ell^p \text{ and } \overline{\operatorname{span}\{e_k \mid k \in \mathbb{Z}_{>0}\}} = \ell^p.$$

## §2.2.1 Examples of linear operators

Example 2.13 Let $A = (a_{ij})$ in $M_{n\times m}(\mathbb{R})$ and define a bounded linear operator

$$\Lambda : \mathbb{R}^m \to \mathbb{R}^n \quad \text{given by} \quad \Lambda(x_1, \dots, x_m) = (y_1, \dots, y_n)$$

where $y_i = \sum_{j=1}^{m} a_{ij} x_j$.

Example 2.14 Let $p \in [1,\infty]_{\mathbb{R}}$ and $(\lambda_1, \lambda_2, \dots)$ a sequence in $\mathbb{R}$ and define a linear operator

$$\Lambda : \ell^p \to \ell^p \quad \text{by} \quad \Lambda(x_1, \dots, x_n) = (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3, \dots)$$

HW Show that ~~if~~ $(\lambda_1, \lambda_2, \dots)$ is bounded ~~then~~ if and only if $\Lambda$ is bounded.

HW Show that $\|\Lambda\| = \sup\{|\lambda_k| \mid k \in \mathbb{Z}_{>0}\}$

Example 2.15 Let $\Lambda f = \frac{d}{dx} f$.

HW Is $\Lambda : C([0,\pi]) \to C([0,\pi])$ a well defined linear operator? Is $\Lambda : BC([0,\pi]) \to BC([0,\pi])$ a well defined linear operator? What is the proper choice of $\mathrm{Dom}(\Lambda)$?

HW Show that $\Lambda : BC([0,\pi]) \to BC([0,\pi])$ is not bounded. (hint: Use $f_k = \sin kx$.)

Example (2.16): Let $p \in [1,\infty)_{\mathbb{R}}$ and $a \in \mathbb{R}$. Define

$$(\Lambda_a f)(x) = f(x-a)$$

HW: Show that $\|\Lambda_a f\|_{L^p} = \|f\|_{L^p}$

HW: Show that $\Lambda_a : L^p \to L^p$ is a bounded linear operator with $\|\Lambda_a\| = 1$. Show that $\Lambda_a$ is injective and surjective.

Example 2.17 Let $p \in [1,\infty]_{\mathbb{R}}$. Define $\Lambda_+ : \ell^p \to \ell^p$ and $\Lambda_- : \ell^p \to \ell^p$ by

$$\Lambda_+(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots) \quad \text{and}$$

$$\Lambda_-(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots).$$

W Show that $\|\Lambda_+\| = \|\Lambda_-\| = 1$. Show that $\Lambda_+$ is injective and not surjective. Show that $\Lambda_-$ is surjective but not injective. Show that $\Lambda_+$ and $\Lambda_-$ are linear continuous operators.

Example 2.18 Let $\Omega \subseteq \mathbb{R}$ and let $g : \Omega \to \mathbb{R}$ be a bounded measurable function. Let $p \in [1,\infty]_{\mathbb{R}}$.

HW. Show that

$M_g : L^p(\Omega) \to L^p(\Omega)$ given by $(M_g f)(x) = g(x) f(x)$.

is a continuous linear operator and

$$\|M_g\| = \|g\|_{L^\infty}$$

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(3)

Example 2.19 Let $a, b \in \mathbb{R}$ with $a < b$.
Define $\Lambda : C^0([a,b]) \to C^0([a,b])$ by

$$(\Lambda f)(x) = \int_a^x f(y)\,dy.$$

HW: Show that $\Lambda$ is a bounded linear operator.

HW: Show that

$$\|\Lambda f\| \le (b-a)\|f\| \quad \text{and} \quad \|\Lambda\| \le (b-a).$$