Math 541
Modern Algebra
A first course in Abstract Algebra
Lecturer: Arun Ram

Fall 2007

University of Wisconsin-Madison
Mathematics Department

Homework 8: Due November 1, 2007

To grade: 4, 6, 11.

1. Let $\mathbb{D}$ be a division ring. Show that the ideals of $\mathbb{D}$ are $\{0\}$ and $\mathbb{D}$.
2. Let $\mathbb{F}$ be a field. Show that the ideals of $M_n(\mathbb{F})$ are $\{0\}$ and $M_n(\mathbb{F})$.
3. Show that each ideal of $\mathbb{Z}$ is generated by one element.
4. Show that each ideal of $\mathbb{R}[x]$ is generated by one element.
5. Give an example of a ring R and an ideal I such that I is not generated by one element (in any possible way). Be sure to *prove* that I is not generated by one element.
6. Show that $(\mathbb{Z}/2\mathbb{Z})\times(\mathbb{Z}/5\mathbb{Z}) \simeq \mathbb{Z}/10\mathbb{Z}$ as groups.
7. Show that the product of groups $(\mathbb{Z}/2\mathbb{Z})\times(\mathbb{Z}/2\mathbb{Z})$ is *not* isomorphic to the group $\mathbb{Z}/4\mathbb{Z}$.
8. Show that $\mathbb{R}[x]/\langle x^2+1\rangle \simeq \mathbb{C}$.
9. Let H be a subgroup of a group G. The *canonical injection* is the map $\iota : H \to G$ given by
$$\begin{array}{cccc}\iota: & H & \longrightarrow & G \\ & h & \mapsto & h\end{array}$$
Show that $\iota : H \to G$ is a well defined injective group homomorphism.
10. Let N be a normal subgroup of a group G. The *canonical surjection* or *canonical projection* is the map $\pi : G \to G/N$ given by

$$\begin{array}{rccc} \pi : & G & \longrightarrow & G/N \\ & g & \mapsto & gN \end{array}$$

Show that $\pi : G \to G/N$ is a well defined surjective group homomorphism and that im $\pi = G/N$ and ker $\pi = N$.

11. Using the notations of problem 10, let M be a subgroup of G. Show that
    1. $M/N = \{mN \mid m \in M\}$ is a subgroup of G/N.
    2. M/N is a normal subgroup of G/N if M is a normal subgroup of G.
    3. $M/N = \pi(M)$ and if M contains N Then $\pi^{-1}(\pi(M)) = M$.
    4. Conclude that *there is a one-to-one correspondence between subgroups of G containing N and subgroups of* G/N.
    5. Show that *this correspondence takes normal subgroups to normal subgroups*.

12. Let I be an ideal of a ring R. The *canonical injection* is the map $\iota : I \to R$ given by

$$\begin{array}{rccc} \iota : & I & \longrightarrow & R \\ & i & \mapsto & i \end{array}$$

Show that $\iota : I \to R$ is a well defined injective ring homomorphism.

13. Let I be an ideal of a ring R. The *canonical surjection* or *canonical projection* is the map $\pi : R \to R/I$ given by

$$\begin{array}{rccc} \pi : & R & \longrightarrow & R/I \\ & r & \mapsto & r+I \end{array}$$

Show that $\pi : R \to R/I$ is a well defined surjective homomorphism and that im $\pi = R/I$ and ker $\pi = I$.

14. Using the notations of problem 13, let J be an ideal of R. Show that
    1. $J/I = \{j + I \mid j \in I\}$ is an ideal of R/I.
    2. $J/I = \pi(J)$ and if J contains I then $\pi^{-1}(\pi(J)) = J$.
    3. Conclude that *there is a one-to-one correspondence between ideals of R containing I and ideals of* R/I.