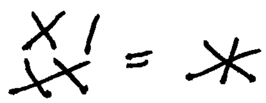
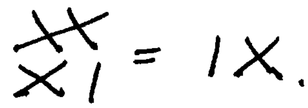
HW3 Problem 1

An <u>abelian group</u> is a group G such that if $g_1, g_2 \in G$ then $g_1 g_2 = g_2 g_1$

The group $\mathbb{Z}/38\mathbb{Z}$ is abelian.

The group S_3 is not abelian since

 and .

HW3 Problem 2

An <u>$n \times n$ permutation matrix</u> is an $n \times n$ matrix such that there is exactly one 1 in each row and each column and all other entries are 0.

The <u>symmetric group</u> S_n is the set of $n \times n$ permutation matrices with operation matrix multiplication.

HW3 Problem 3

A *cyclic group* is a group generated by one element.

The cyclic groups are

$$C_n = \{1, g, g^2, \ldots, g^{n-1}\} \text{ with } g^n = 1$$

and

$$C_\infty = \{\ldots, g^{-2}, g^{-1}, 1, g, g^2, \ldots\}.$$

Note that $C_n \cong \mathbb{Z}/n\mathbb{Z}$ and $C_\infty \cong \mathbb{Z}$.

HW3 Problem 4 Let $n \in \mathbb{Z}_{>0}$.

The *dihedral group* D_n is the group generated by a and b with operation determined by

$$a^n = 1, \; b^2 = 1 \text{ and } ba = a^{n-1}b.$$

HW3 Problem 5

The *Klein 4 group* is the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

HW3 Problem 6

Let G and H be groups.

The *product* of G and H is the set

$$G \times H = \{(g, h) \mid g \in G, h \in H\}$$

with operation given by

$$(g_1,h_1)(g_2,h_2)=(g_1g_2,h_1h_2).$$

If $H=\{1\}$ then $G\times\{1\}\cong G$.

Similarly, if $G=\{1\}$ then $\{1\}\times H\cong H$.

So every group G is a product, $G\times\{1\}$.

The smallest example which is not of this form is the

Klein 4 group, $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$.

<u>HW3 Problem 8</u> See HW3 Problem 2.

<u>HW3 Problem 10</u>

Let G be a group.
A <u>subgroup</u> of G is a subset H of G such that

(a) If $h_1,h_2\in H$ then $h_1h_2\in H$

(b) $1\in H$

(c) If $h\in H$ then $h^{-1}\in H$.

Some examples are

$$SL_n(\mathbb{C})\subseteq GL_n(\mathbb{C}),\quad S_n\subseteq O_n(\mathbb{C}),\quad \mathbb{Z}\subseteq\mathbb{Q},$$

$$\mathbb{Z}/2\mathbb{Z}\times\{0\}\subseteq\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}.$$

Another example is the subgroup

$$\{|||, \text{X}, \text{XX}\} \subseteq S_3.$$

Here, $\{|||, \text{X}, \text{XX}\} \cong \mathbb{Z}/3\mathbb{Z}$.

<u>HW3 Problem 11</u>

Let S and T be sets

The sets S and T have equal <u>cardinality</u> if there exists a bijection $\varphi : S \to T$.

Write $\mathrm{Card}(S) = \mathrm{Card}(T)$ if S and T have equal cardinality.

<u>HW3 Problem 12</u>

Let S be a set.

The set S is <u>finite</u> if $\mathrm{Card}(S) = \mathrm{Card}(\mathbb{Z}/n\mathbb{Z})$ for some $n \in \mathbb{Z}_{>0}$.

The set S is <u>countable</u> if S is finite or $\mathrm{Card}(S) = \mathrm{Card}(\mathbb{Z})$

The set S is <u>infinite</u> if S is not finite.

The set S is <u>uncountable</u> if S is not countable.

HW3 Problem 13

Show that $\mathrm{Card}(\mathbb{Z}_{>0}) = \mathrm{Card}(\mathbb{Z}_{\geq 0})$.

Proof

To show: $\mathrm{Card}(\mathbb{Z}_{>0}) = \mathrm{Card}(\mathbb{Z}_{\geq 0})$

To show: There exists a bijection $\varphi : \mathbb{Z}_{>0} \to \mathbb{Z}_{\geq 0}$.

Let $\varphi : \mathbb{Z}_{>0} \to \mathbb{Z}_{\geq 0}$ be given by

$$\cancel{\varphi(i) \mapsto}\ \varphi(i) = i-1.$$

To show: φ is a bijection.

To show: The inverse function to φ exists.

Let $\varphi^{-1} : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{>0}$ be given by

$$\varphi^{-1}(j) = j+1.$$

To show: (a) $\varphi \circ \varphi^{-1} = \iota_{\mathbb{Z}_{\geq 0}}$

(b) $\varphi^{-1} \circ \varphi = \iota_{\mathbb{Z}_{>0}}$.

(a) ~~Assume~~ ~~If~~ ~~$j \in \mathbb{Z}_{\geq 0}$. Then~~ To show: If $j \in \mathbb{Z}_{\geq 0}$ then $(\varphi \circ \varphi^{-1})(j) = \iota_{\mathbb{Z}_{\geq 0}}(j)$.

Assume $j \in \mathbb{Z}_{\geq 0}$.

To show: $(\varphi \circ \varphi^{-1})(j) = \iota_{\mathbb{Z}_{\geq 0}}(j)$.

$$(\varphi \circ \varphi^{-1})(j) = \varphi(\varphi^{-1}(j)) = \varphi(j+1) = j+1-1 = j = \iota_{\mathbb{Z}_{\geq 0}}(j).$$

So $(\varphi \circ \varphi^{-1})(j) = \iota_{\mathbb{Z}_{\geq 0}}(j)$.

So $\varphi \circ \varphi^{-1} = \iota_{\mathbb{Z}_{\geq 0}}$.

(b) To show: $(\varphi^{-1} \circ \varphi) = \iota_{\mathbb{Z}_{\geq 0}}$.

To show: If $i \in \mathbb{Z}_{\geq 0}$ then $(\varphi^{-1} \circ \varphi)(i) = \iota_{\mathbb{Z}_{\geq 0}}(i)$

Assume $i \in \mathbb{Z}_{\geq 0}$.

To show: $(\varphi^{-1} \circ \varphi)(i) = \iota_{\mathbb{Z}_{\geq 0}}(i)$.

$$(\varphi^{-1} \circ \varphi)(i) = \varphi^{-1}(\varphi(i)) = \varphi^{-1}(i-1) = i-1+1 = i = \iota_{\mathbb{Z}_{\geq 0}}(i).$$

So $(\varphi^{-1} \circ \varphi)(i) = \iota_{\mathbb{Z}_{\geq 0}}(i)$

So $\varphi^{-1} \circ \varphi = \iota_{\mathbb{Z}_{\geq 0}}$.

So the inverse to φ exists.

So φ is a bijection

So $\mathrm{Card}(\mathbb{Z}_{\geq 0}) = \mathrm{Card}(\mathbb{Z}_{>0})$.

<u>HW3 Problem 18</u> Let G and H be groups

A group <u>homomorphism</u> from G to H is a function $\varphi: G \to H$ such that

(a) If $g_1, g_2 \in G$ then $\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2)$

(b) $\varphi(1) = 1$,

(c) If $g \in G$ then $\varphi(g^{-1}) = \varphi(g)^{-1}$.

An <u>isomorphism</u> between G and H is a homomorphism $\varphi: G \to H$ such that

(a) $\varphi^{-1}: H \to G$ exists

(b) φ^{-1} is a group homomorphism.

The function

$$\det: GL_n(\mathbb{C}) \to GL_1(\mathbb{C})$$

is a homomorphism.

The ~~function~~ homomorphism $\varphi: S_3 \to D_3$ ~~given~~ determined by

$$\varphi(\times\!\!\times) = a \quad \text{and} \quad \varphi(\times|) = b$$

is an isomorphism.

<u>HW3 Problem 21</u>

Let $\varphi: G \to H$ be a group homomorphism.

The <u>kernel</u> of φ is

$$\ker \varphi = \{ g \in G \mid \varphi(g) = 1 \}$$

The <u>image</u> of φ is

$$\operatorname{im} \varphi = \{ \varphi(g) \mid g \in G \}.$$

To show: (a) $\ker \varphi$ is a subgroup of G

(b) $\operatorname{im} \varphi$ is a subgroup of H.

(a) To show: (aa) If $g_1, g_2 \in \ker\varphi$ then $g_1 g_2 \in \ker\varphi$

(ab) $1 \in \ker\varphi$

(ac) If $g \in \ker\varphi$ then $g^{-1} \in \ker\varphi$.

(aa) Assume $g_1, g_2 \in \ker\varphi$.

To show: $g_1 g_2 \in \ker\varphi$.

We know $\varphi(g_1) = 1$ and $\varphi(g_2) = 1$.

To show: $\varphi(g_1 g_2) = 1$.

$$\begin{aligned}\varphi(g_1 g_2) &= \varphi(g_1)\varphi(g_2), \text{ since } \varphi \text{ is a homomorphism,} \\ &= 1 \cdot 1 = \underline{1}.\end{aligned}$$

So $g_1 g_2 \in \ker\varphi$.

(ab) To show: $1 \in \ker\varphi$.

To show: $\varphi(1) = 1$.

This follows from condition (b) in the definition of homomorphism.

(ac) To show: If $g \in \ker\varphi$ then $g^{-1} \in \ker\varphi$.

Assume $g \in \ker\varphi$.

To show: $g^{-1} \in \ker\varphi$.

We know: $\varphi(g) = 1$.

To show: $\varphi(g^{-1})=1$.

$$\begin{aligned}\varphi(g^{-1}) &= \varphi(g)^{-1}, \text{ since } \varphi \text{ is a homomorphism}\\ &= 1^{-1}, \text{ since } g\in\ker\varphi,\\ &= 1, \text{ since } 1\cdot 1=1.\end{aligned}$$

So $g^{-1}\in\ker\varphi$.

So $\ker\varphi$ is a subgroup of G.

(b) To show: $\operatorname{im}\varphi$ is a subgroup of H.

To show: (ba) If $h_1,h_2\in\operatorname{im}\varphi$ then ~~$\varphi(h_1$~~ $h_1h_2\in\operatorname{im}\varphi$.

(bb) $1\in\operatorname{im}\varphi$.

(bc) If $h\in\operatorname{im}\varphi$ then $h^{-1}\in\operatorname{im}\varphi$.

(ba) Assume $h_1,h_2\in\operatorname{im}\varphi$.

To show: $h_1h_2\in\operatorname{im}\varphi$.

We know that there exist $g_1,g_2\in G$ such that $\varphi(g_1)=h_1$ and $\varphi(g_2)=h_2$.

To show: There exists $k\in G$ such that $\varphi(k)=h_1h_2$.

Let $k=g_1g_2$.

To show: $\varphi(k)=h_1h_2$

$$\varphi(k)=\varphi(g_1g_2)=\varphi(g_1)\varphi(g_2)=h_1h_2$$

So $h_1h_2\in\operatorname{im}\varphi$.

(bb) To show: $1 \in \operatorname{im}\varphi$.

To show: There exists $k \in G$ such that $\varphi(k) = 1$.

Let $k = 1$.

To show: $\varphi(k) = 1$

$\varphi(k) = \varphi(1) = 1$, since φ is a homomorphism.

So $1 \in \operatorname{im}\varphi$.

(bc) To show: If $h \in \operatorname{im}\varphi$ then $h^{-1} \in \operatorname{im}\varphi$.

Assume $h \in \operatorname{im}\varphi$

To show: $h^{-1} \in \operatorname{im}\varphi$.

We know that there exists $g \in G$ such that $\varphi(g) = h$.

To show: There exists $k \in G$ such that $\varphi(k) = h^{-1}$.

Let $k = g^{-1}$.

To show: $\varphi(k) = h^{-1}$.

$\varphi(k) = \varphi(g^{-1}) = \varphi(g)^{-1}$, since φ is a homomorphism,

$= h^{-1}$, since $\varphi(g) = h$.

So $h^{-1} \in \operatorname{im}\varphi$.

So $\operatorname{im}\varphi$ is a subgroup of H. //

HW3 Problem 22

Let G and H be groups and let $\varphi: G \to H$ be a function such that

if $g_1, g_2 \in G$ then $\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2)$.

Show that $\varphi(1) = 1$.

Proof Assume G and H are groups and $\varphi: G \to H$ is a function such that if $g_1, g_2 \in G$ then $\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2)$.

To show: $\varphi(1) = 1$.

~~Case Let $g \in G$. Then~~

$\cancel{\varphi(1) = \varphi(g g^{-1}) = \varphi(g}$

$\cancel{\varphi(1) = \varphi(1) \cdot \varphi(1}$

$$\varphi(1) = \varphi(1 \cdot 1) = \varphi(1) \cdot \varphi(1).$$

Since H is a group $\varphi(1)^{-1} \in H$. Multiply on the right by $\varphi(1)^{-1}$ to get

$$\varphi(1)\varphi(1)^{-1} = \varphi(1) \cdot \varphi(1)\varphi(1)^{-1}$$

So

$$1 = \varphi(1) \cdot 1 = \varphi(1).$$

So $\varphi(1) = 1$.

HW3 Problem 23

Let G and H be groups and let $\varphi: G \to H$ be a function such that

if $g_1, g_2 \in G$ then $\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2)$.

Show that if $g \in G$ then $\varphi(g^{-1}) = \varphi(g)^{-1}$.

Proof Assume G and H are groups and $\varphi: G \to H$ is a function such that if $g_1, g_2 \in G$ then

$$\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2).$$

To show: If $g \in G$ then $\varphi(g^{-1}) = \varphi(g)^{-1}$.

Assume $g \in G$

To show: $\varphi(g^{-1}) = \varphi(g)^{-1}$.

To show: $\varphi(g^{-1})$ is the inverse of $\varphi(g)$.

To show: (a) $\varphi(g^{-1})\varphi(g) = 1$

(b) $\varphi(g)\varphi(g^{-1}) = 1$.

(a) $\varphi(g^{-1})\varphi(g) = \varphi(g^{-1}g) = \varphi(1) = 1$, by Problem 22

(b) $\varphi(g)\varphi(g^{-1}) = \varphi(gg^{-1}) = \varphi(1) = 1$, by Problem 22.

So $\varphi(g^{-1})$ is the inverse of $\varphi(g)$.

So $\varphi(g^{-1}) = \varphi(g)^{-1}$ //.

<u>HW3 Problem 24</u> Let φ be a field homomorphism. Show that φ is injective.

<u>Proof</u>:

Assume $\mathbb{F}$ and $\mathbb{K}$ are fields and $\varphi : \mathbb{F} \to \mathbb{K}$ is a field homomorphism.

To show: $\varphi : \mathbb{F} \to \mathbb{K}$ is injective.

To show: If $c_1, c_2 \in \mathbb{F}$ and $\varphi(c_1) = \varphi(c_2)$ then $c_1 = c_2$.

Assume $c_1, c_2 \in \mathbb{F}$ and $\varphi(c_1) = \varphi(c_2)$

To show: $c_1 = c_2$.

Proof by contradiction.

Assume $c_1 \neq c_2$.

Then $c_1 - c_2 \neq 0$ and $(c_1 - c_2)^{-1} \in \mathbb{F}$.

So
$$\begin{aligned} 1 = \varphi(1) &= \varphi((c_1-c_2)(c_1-c_2)^{-1}) \\ &= \varphi(c_1-c_2)\,\varphi((c_1-c_2)^{-1}) \\ &= 0 \cdot \varphi((c_1-c_2)^{-1}) = 0. \end{aligned}$$

So $1 = 0$ in $\mathbb{K}$.

So, if $k \in \mathbb{K}$ then

$k = k \cdot 1 = k \cdot 0 = 0$.

So $\mathbb{K} = \{0\}$.

At this point we see that the problem is not stated quite right. It should say:

Let $\varphi: \mathbb{F} \to \mathbb{K}$ be a field homorphism such that $\mathbb{K}$ is not the zero field. Then φ is injective.

Proof

Assume $\mathbb{F}$ and $\mathbb{K}$ are fields, $\mathbb{K} \neq \{0\}$ and $\varphi: \mathbb{F} \to \mathbb{K}$ is a field homomorphism.

To show: $\varphi: \mathbb{F} \to \mathbb{K}$ is injective.

To show: If $c_1, c_2 \in \mathbb{F}$ and $\varphi(c_1) = \varphi(c_2)$ then $c_1 = c_2$.

Assume $c_1, c_2 \in \mathbb{F}$ and $\varphi(c_1) = \varphi(c_2)$.

To show: $c_1 = c_2$.

Proof by contradiction.

Assume $c_1 \neq c_2$.

Then $c_1 - c_2 \neq 0$ and $(c_1 - c_2)^{-1} \in \mathbb{F}$.

So
$$\begin{aligned} 1 = \varphi(1) &= \varphi((c_1-c_2)(c_1-c_2)^{-1}) \\ &= \varphi(c_1-c_2)\varphi((c_1-c_2)^{-1}) \\ &= 0 \cdot \varphi((c_1-c_2)^{-1}) = 0. \end{aligned}$$

So $1 = 0$ in $\mathbb{K}$.

So, if $k \in \mathbb{K}$ then $k = k \cdot 1 = k \cdot 0 = 0$.

So $\mathbb{K} = \{0\}$.

Contradiction to $\mathbb{K} \neq \{0\}$.

So $c_1 = c_2$. So φ is injective.

HW3 Problem 7

Let $M_n(\mathbb{C})$ be the set of $n \times n$ matrices with entries in $\mathbb{C}$.

The general linear group is the set

$$GL_n(\mathbb{C}) = \{ g \in M_n(\mathbb{C}) \mid g \text{ is invertible} \}$$

with operation matrix multiplication

The special linear group is the subgroup

$$SL_n(\mathbb{C}) = \{ g \in GL_n(\mathbb{C}) \mid \det(g) = 1 \}$$

of $GL_n(\mathbb{C})$.

The orthogonal group is the subgroup,

$$O_n(\mathbb{C}) = \{ g \in GL_n(\mathbb{C}) \mid g g^t = 1 \},$$

of $GL_n(\mathbb{C})$.

The special orthogonal group is

$$SO_n(\mathbb{C}) = O_n(\mathbb{C}) \cap SL_n(\mathbb{C})$$

If $g = (g_{ij})$ is an $n \times n$ matrix

$\bar{g} = (\bar{g}_{ij})$ where $\bar{g}_{ij}$ is the complex conjugate of g_{ij} (If $a+bi \in \mathbb{C}$ then $\overline{a+bi} = a-bi$).

The <u>unitary group</u> is

$$U_n(\mathbb{C}) = \{ g \in GL_n(\mathbb{C}) \mid g\bar{g}^t = 1 \}$$

with operation matrix multiplication.

The <u>special</u> <u>unitary group</u> is

$$SU_n(\mathbb{C}) = U_n(\mathbb{C}) \cap SL_n(\mathbb{C})$$

The <u>symplectic group</u> is

$$Sp_{2n}(\mathbb{C}) = \{ g \in GL_{2n}(\mathbb{C}) \mid gJg^t = J \}$$

where J is the matrix

$$J = \begin{array}{cl} \begin{array}{cc} \overbrace{\hphantom{xxxxxxx}}^{n} & \overbrace{\hphantom{xxxxxxx}}^{n} \end{array} & \\ \left(\begin{array}{c|c} 0 & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \\ \hline \begin{matrix} -1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{matrix} & 0 \end{array}\right) & \begin{array}{l} \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} n \\ \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} n \end{array} \end{array}$$

HW3 Problem 9

The <u>general linear group</u> $GL_n(\mathbb{C})$ acts on the n-dimensional vector space $\mathbb{C}^n = M_{n\times 1}(\mathbb{C})$ of $n\times 1$ matrices by matrix multiplication.

The symmetric group S_n, of $n\times n$ permutation matrices acts on the n-element set

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\}$$

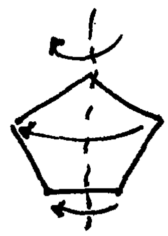
by matrix multiplication.

The dihedral group D_n, generated by a and b with $a^n=1$, $b^2=1$ and $ba = a^{n-1}b$, acts on a regular n-gon by having

a act by rotation by $\frac{2\pi}{n}$

and b act by reflection in the axis determined by the center and a vertex.

For example D_5 acts on a pentagon by

a  and b 

Since the cyclic group

$$C_n = \{1, a, a^2, \ldots, a^{n-1}\}$$

is a subgroup of D_n, it also acts on the n-gon.

HW3 Problem 19

Let R and S be rings

A <u>ring homomorphism</u> from R to S is a function $\varphi: R \to S$ such that

(a) If $r_1, r_2 \in R$ then $\varphi(r_1 + r_2) = \varphi(r_1) + \varphi(r_2)$,

(b) $\varphi(0) = 0$,

(c) If $r \in R$ then $\varphi(-r) = -\varphi(r)$,

(d) If $r_1, r_2 \in R$ then $\varphi(r_1 r_2) = \varphi(r_1)\varphi(r_2)$,

(e) $\varphi(1) = 1$.

An <u>isomorphism</u> between R and S is a function $\varphi: R \to S$ such that

(a) $\varphi^{-1}: S \to R$ exists,

(b) φ is a ring homomorphism,

(c) φ^{-1} is a ring homomorphism.

HW3 Problem 20

Let $\mathbb{F}$ and $\mathbb{K}$ be fields

A field homomorphism from $\mathbb{F}$ to $\mathbb{K}$ is a function $\varphi : \mathbb{F}$ to $\mathbb{K}$ such that

(a) If $r_1, r_2 \in \mathbb{F}$ then $\varphi(r_1 + r_2) = \varphi(r_1) + \varphi(r_2)$,

(b) $\varphi(0) = 0$,

(c) If $r \in \mathbb{F}$ then $\varphi(-r) = -\varphi(r)$,

(d) If $r_1, r_2 \in \mathbb{F}$ then $\varphi(r_1 r_2) = \varphi(r_1)\varphi(r_2)$

(e) $\varphi(1) = 1$

(f) If $r \in \mathbb{F}$ and $r \neq 0$ then $\varphi(r^{-1}) = \varphi(r)^{-1}$.

An isomorphism between $\mathbb{F}$ and $\mathbb{K}$ is a function $\varphi : \mathbb{F} \to \mathbb{K}$ such that

(a) $\varphi^{-1} : \mathbb{K} \to \mathbb{F}$ exists,

(b) φ is a field homomorphism,

(c) φ^{-1} is a field homomorphism.

<u>HW3 Problem 17</u> Show that $\mathrm{Card}(\mathbb{R}) = \mathrm{Card}(\mathbb{C})$.

<u>Proof</u> To show: $\mathrm{Card}(\mathbb{C}) = \mathrm{Card}(\mathbb{R})$.

To show: There exists a bijection $\varphi : \mathbb{C} \to \mathbb{R}$

Let

$$\varphi(a+bi) = b_\ell a_\ell \cdots b_2 a_2 b_1 a_1 b_0 a_0 . a_{-1} b_{-1} a_{-2} b_{-2} \cdots$$

if the decimal expansions of a and b are

$$a_\ell \cdots a_1 a_0 . a_{-1} a_{-2} \cdots \quad \text{and} \quad b_\ell \cdots b_1 b_0 . b_{-1} b_{-2} \cdots .$$

To show: φ is bijective

To show: $\varphi^{-1} : \mathbb{R} \to \mathbb{C}$ exists.

Let

$$\varphi^{-1}(c_{2\ell-1} \cdots c_1 c_0 . c_{-1} c_{-2} \cdots)$$

$$= c_{2\ell-2} \cdots c_2 c_0 . c_{-1} c_{-3} \cdots + c_{2\ell-1} \cdots c_3 c_1 . c_{-2} c_{-4} \cdots i$$

To show: φ^{-1} is the inverse of φ.

To show: (a) $\varphi \circ \varphi^{-1} = \iota_{\mathbb{R}}$

(b) $\varphi^{-1} \circ \varphi = \iota_{\mathbb{C}}$.

(a) To show: If $x \in \mathbb{R}$ then $(\varphi \circ \varphi^{-1})(x) = \iota_{\mathbb{R}}(x)$.

Assume $x \in \mathbb{R}$, $x = x_{2\ell-1} \cdots x_1 x_0 . x_{-1} x_{-2} \cdots$

Then

$$(\varphi \circ \varphi^{-1})(x) = \varphi(\varphi^{-1}(x)) = \varphi\begin{pmatrix} x_{2\ell-2} \cdots x_2 x_0 . x_{-1} x_{-3} \cdots \\ + x_{2\ell-1} \cdots x_3 x_1 . x_{-2} x_{-3} \cdots i \end{pmatrix}$$

$= x_{2\ell-1}x_{2\ell-2}\cdots x_3x_2x_1x_0.x_{-1}x_{-2}x_{-3}\ldots = x = \iota_{\mathbb{R}}(x).$

(b) To show: If $a+bi\in\mathbb{C}$ then $(\varphi^{-1}\circ\varphi)(a+bi) = \iota_{\mathbb{C}}(a+bi)$.

Assume $a+bi\in\mathbb{C}$,

$$a = a_\ell\cdots a_1a_0.a_{-1}a_{-2}\cdots \quad \text{and}$$
$$b = b_\ell\cdots b_1b_0.b_{-1}b_{-2}\cdots.$$

Then

$$(\varphi^{-1}\circ\varphi)(a+bi) = \varphi^{-1}(\varphi(a+bi))$$
$$= \varphi^{-1}(b_\ell a_\ell\cdots b_1a_1b_0a_0.a_{-1}b_{-1}a_{-2}b_{-2}\cdots)$$
$$= a_\ell\cdots a_1a_0.a_{-1}a_{-2}\ldots + b_\ell b_{\ell-1}\cdots b_1b_0.b_{-1}b_{-2}\cdots i$$
$$= a+bi = \iota_{\mathbb{C}}(a+bi).$$

So φ is invertible.

So $\mathrm{Card}(\mathbb{R}) = \mathrm{Card}(\mathbb{C})$.

HW3 Problem 14 Show that $\mathrm{Card}(\mathbb{Z}_{\geq 0}) = \mathrm{Card}(\mathbb{Z})$

To show: $\mathrm{Card}(\mathbb{Z}) = \mathrm{Card}(\mathbb{Z}_{\geq 0})$

To show: There exists a bijection $\varphi: \mathbb{Z} \to \mathbb{Z}_{\geq 0}$.

Let $\varphi: \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ be given by

$$\varphi(i) = \begin{cases} 0, & \text{if } i = 0, \\ 2i, & \text{if } i \in \mathbb{Z}_{>0}, \\ -2i-1, & \text{if } -i \in \mathbb{Z}_{>0}. \end{cases}$$

To show: φ is bijective.

To show: $\varphi^{-1}: \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ exists.

Let $\varphi^{-1}: \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ be given by

$$\varphi^{-1}(j) = \begin{cases} j/2, & \text{if } j \text{ is even}, \\ -\frac{j+1}{2}, & \text{if } j \text{ is odd}. \end{cases}$$

To show: (a) $\varphi \circ \varphi^{-1} = \iota_{\mathbb{Z}_{\geq 0}}$

(b) $\varphi^{-1} \circ \varphi = \iota_{\mathbb{Z}}$.

(a) To show: If $j \in \mathbb{Z}_{\geq 0}$ then $(\varphi \circ \varphi^{-1})(j) = \iota_{\mathbb{Z}_{\geq 0}}(j)$.

Assume $j \in \mathbb{Z}_{\geq 0}$

To show: $(\varphi \circ \varphi^{-1})(j) = \iota_{\mathbb{Z}_{\geq 0}}(j)$

<u>Case 1</u> j is even.

$$(\varphi \circ \varphi^{-1})(j) = \varphi(j/2) = 2 \cdot \frac{j}{2} = j = \iota_{\mathbb{Z}_{\geq 0}}(j).$$

Case 2 j is odd.

$$(\varphi\circ\varphi^{-1})(j) = \varphi(\varphi^{-1}(j)) = \varphi\left(-\frac{j+1}{2}\right) = -2\left(-\frac{j+1}{2}-1\right)$$

$$= j-1+1 = j.$$

(d) To show: $\varphi^{-1}\circ\varphi = \iota_{\mathbb{Z}}$

To show: If $i\in\mathbb{Z}$ then $(\varphi^{-1}\circ\varphi)(i) = \iota_{\mathbb{Z}}(i)$

Assume $i\in\mathbb{Z}$.

To show $(\varphi^{-1}\circ\varphi)(i) = \iota_{\mathbb{Z}}(i)$.

Case 1: $i\in\mathbb{Z}_{>0}$.

$$(\varphi^{-1}\circ\varphi)(i) = \varphi^{-1}(\varphi(i)) = \varphi^{-1}(2i) = \frac{2i}{2} = i.$$

Case 2: $i\in\mathbb{Z}_{<0}$

$$(\varphi^{-1}\circ\varphi)(i) = \varphi^{-1}(\varphi(i)) = \varphi^{-1}(-2i-1)$$

$$= -\frac{-2i-1+1}{2} = i.$$

Case 3: $i=0$

$$(\varphi^{-1}\circ\varphi)(0) = \varphi^{-1}(\varphi(0)) = \varphi^{-1}(0) = 0/2.$$

So φ is invertible

So φ is a bijection

So $\mathrm{Card}(\mathbb{Z}) = \mathrm{Card}(\mathbb{Z}_{\geq 0})$.