THE UNIVERSITY of WISCONSIN MADISON

University of Wisconsin-Madison
Mathematics Department

Math 541
Modern Algebra
A first course in Abstract Algebra
Lecturer: Arun Ram

Fall 2007

Homework 2: Due September 19, 2007

1. Define monoid without identity, monoid, group, ring without identity, ring, division ring and field, and give examples. Make sure that your example of a monoid without identity is not a monoid, that your example of a monoid is not a group, etc.
2. Give and example of an operation on $\mathbb{Z}$ that is not associative.
3. Let G be a group. Show that the identity element of G is unique.
4. Let G be a group and let $g \in G$. Show that the inverse of g is unique.
5. Why isn't $\{0, 1, 2, 3, 4, 5\}$ a group?
6. Show that $-(-5) = 5$.
7. Show that $1/(1/5) = 5$.
8. Show that $-1 \cdot 5 = -5$.
9. Show that $0 \cdot 5 = 0$.
10. Define $\mathbb{Q}$ and prove that it is a field.
11. Define the quaternions and show that they are a division ring and not a field.
12. Define $\mathbb{Z}/n\mathbb{Z}$ and prove that it is a group.
13. Define $\mathbb{Z}/n\mathbb{Z}$ and prove that it is a ring.
14. For which positive integers n is $\mathbb{Z}/n\mathbb{Z}$ a field?
15. Let n be a positive integer. An *nth root of unity* is a complex number a such that $a^n = 1$.

Let C_n be the set of nth roots of unity in $\mathbb{C}$. Determine C_3, C_4, C_5, and graph these sets.

16. Let C_n be the set of nth roots of unity in $\mathbb{C}$. Show that C_n is a group.

17. Define $M_n(\mathbb{C})$ and prove that it is a ring.

18. Define $\mathbb{C}[x]$ and prove that it is a ring.

19. Define $\mathbb{C}(x)$ and prove that it is a field.

20. Define $\mathbb{C}[[x]]$ and prove that it is a ring.

21. Define $\mathbb{C}((x))$ and prove that it is a field.

22. Show that each element of $\mathbb{C}((x))$ has a unique expression in the form $x^{\ell} p$, where $p \in \mathbb{C}[x]$ and has nonzero constant term.

23. Show that there exists a field with 4 elements.