Math 541 — Homework 1 solutions

[1] • A <u>set</u> is a collection of objects which are called elements.

- The <u>empty set</u> ϕ is the set with no elements.
- A <u>subset</u> T of a set S is a set such that if $t \in T$ then $t \in S$. We write $T \subseteq S$.
- Two sets S and T are <u>equal</u> if $S \subseteq T$ and $T \subseteq S$. We write $S = T$.
- Let S and T be sets. The <u>union of S and T</u> is the set $S \cup T$ of all u such that $u \in S$ or $u \in T$.
  <u>ie</u> $S \cup T = \{u \mid u \in S \text{ or } u \in T\}$.
- Let S and T be sets. The <u>intersection of S and T</u> is the set $S \cap T$ of all u such that $u \in S$ and $u \in T$.
  <u>ie</u> $S \cap T = \{u \mid u \in S \text{ and } u \in T\}$.
- Let S and T be sets. The <u>product of S and T</u> is the set $S \times T$ of all ordered pairs (s,t) where $s \in S$ and $t \in T$.
  <u>ie</u> $S \times T = \{(s,t) \mid s \in S, t \in T\}$.
  More generally, given sets $S_1, \dots, S_n$ the product $\prod_{i=1}^{n} S_i$ is the set of all tuples $(s_1, \dots, s_n)$ such that $s_i \in S_i$.

$\boxed{1}$ (cont'd.)

- <u>Examples</u> -

Let $S = \{-3,-2,-1,0,1,2,3\}$

$T = \{$Stars, Tori, The Shins, Kiss$\}$

$U = \{x \mid x$ is a band or musician that I love$\}$

$V = \{\ddot{\smile}, \ddot{\frown}, \ddot{\smile}, \ddot{-}, \ddot{*}, \ddot{o}\}$

$W = \{a\}$

$X = \{$people I've dated$\}$

$Y = \{$Drew, my mom, Robyn$\}$

$Z = \{$people I love$\}$

$L = \{\frac{1}{n} \mid n \in \mathbb{Z}_{>0}\}$

$M = \{$people who live in Madison$\}$

$N = \{x \in \mathbb{C} \mid x^2+1=0\}$

$O = \{\pm i\}$

Then all of the above are sets. Further:

- $Y \subseteq Z$
- $N = O$
- $S \cup W = \{a,-3,-2,-1,0,1,2,3\}$,
  $T \cup U = \{x \mid x$ is a band or musician I love or $x =$ The Shins or $x =$ Kiss$\}$
- $X \cap Y = \{$Drew, Robyn$\}$,
  $M \cap Y = \emptyset$, $L \cap S = \{1\}$
- $W \times V = \{(a,\ddot{\smile}),(a,\ddot{\frown}),(a,\ddot{\smile}),(a,\ddot{-}),(a,\ddot{*}),(a,\ddot{o})\}$
  $P \times N \times W = \{(i,i,a),(i,-i,a),(-i,i,a),(-i,-i,a)\}$
  $\mathbb{R} \times \mathbb{R} = \{(a,b) \mid a,b \in \mathbb{R}\} = \mathbb{R}^2$.

$\boxed{2}$ Let S, T, and U be sets.

- A <u>function</u> $f: S \to T$ is given by associating to each element $s \in S$ a unique element $f(s) \in T$. We write
$$f: S \to T$$
$$s \mapsto f(s)$$

- A function $f: S \to T$ is <u>injective</u> if it satisfies: if $s_1, s_2 \in S$ and $f(s_1) = f(s_2)$, then $s_1 = s_2$.

- A function $f: S \to T$ is <u>surjective</u> if for each element $t \in T$ there exists $s \in S$ such that $f(s) = t$.

- A function $f: S \to T$ is <u>bijective</u> if it is both injective and surjective.

- Let $f: S \to T$ and $g: T \to U$ be functions. The <u>composition of f and g</u> is the function $g \circ f$ given by
$$(g \circ f): S \to U$$
$$s \mapsto g(f(s)).$$

- The <u>identity map on S</u> is the function ι_S given by:
$$\iota_S: S \to S$$
$$s \mapsto s.$$

- Let $f: S \to T$ be a function. An <u>inverse function to f</u> is a function $f^{-1}: T \to S$ such that

$\boxed{2}$ (cont'd.) $f \circ f^{-1} = \iota_T$ and
$f^{-1} \circ f = \iota_S$.

- Examples - (Using the sets defined in $\boxed{1}$)

Let $f : S \to L$ be given by
$s \mapsto 1$.

Let $g : \mathbb{C} \to \mathbb{C}$ be given by
$z \mapsto \bar{z}$

ie $(a+bi) \mapsto (a-bi)$

Let $h : \mathbb{C} \to \mathbb{R}$ be given by
$a+bi \mapsto b$

Let $k : S \to L$ be given by
$s \mapsto \frac{1}{|s|}$ for $s \neq 0$ and
$0 \mapsto 1$

Let $l : \mathbb{R} \to \mathbb{R}$ be given by
$x \mapsto x^3$

All of the above are well-defined functions.

- l and g are injective functions, whereas the others are not injective.
- g, h, and l are surjective.
- l & g are the only bijective function
- The composition of g and h is given by
$$(h \circ g) : \mathbb{C} \to \mathbb{R}$$
$$(a+bi) \mapsto -b$$
- The inverse function of l is given by
$$l^{-1} : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto x^{1/3}.$$

$\boxed{3}$ Let S be a set.

- a *relation* on S is a subset of $S \times S$. We write $s_1 \sim s_2$ if the pair (s_1, s_2) is in this subset.
- a relation $\sim$ on S is *reflexive* if for each $s \in S$, $s \sim s$.
- a relation $\sim$ on S is *symmetric* if $s_1 \sim s_2$ if and only if $s_2 \sim s_1$.
- a relation $\sim$ on S is *transitive* if it satisfies: if $s_1 \sim s_2$ and $s_2 \sim s_3$ then $s_1 \sim s_3$.
- an *equivalence relation* on S is a relation on S that is reflexive, symmetric, and transitive.
- a relation $\sim$ on S is *antisymmetric* if it satisfies: if $s_1 \sim s_2$ and $s_2 \sim s_1$, then $s_1 = s_2$.
- a ~~[illegible]~~ *partial order* on S is a relation on S that is reflexive, antisymmetric, and transitive.
- a *poset* is a ~~Set~~ S together with a partial order on S.

- *Examples* - (using sets defined in $\boxed{1}$ again)

Let $R_1 = \{$(🙂, 🙁), (🙁, 🙂), (😐, 😮), (😮, 😐)$\}$

$R_2 = \{(-3,3), (3,0), (3,-3), (-3,-3), (-3,0), (0,-3), (0,3), (3,3), (0,0)\}$

$\boxed{3}$ (contd.) $R_3 = \{(\text{Drew}, \text{Robyn}), (\text{Robyn}, \text{Drew})\}$ and $R_4 = \{(1,1), (2,6), (6,1), (2,1)\}$.

Then R_1 is a symmetric and transitive relation on V.

Then R_2 is an equivalence relation on S.

Then R_3 is a reflexive relation on Y.

Then R_4 is a transitive relation on $\mathbb{Z}$.

The relation $\leq$ on the set $\mathbb{Z}_{\geq 0}$ is a partial order, so $(\mathbb{Z}_{\geq 0}, \leq)$ is a poset.

$\boxed{4}$ Let S be a set.

- An <u>operation</u> on S is a map $\circ : S \times S \to S$. If $s_1, s_2 \in S$, we write $s_1 \circ s_2$ instead of $\circ((s_1, s_2))$.
- An operation $\circ$ on S is <u>associative</u> if for all $s_1, s_2, s_3 \in S$
$$(s_1 \circ s_2) \circ s_3 = s_1 \circ (s_2 \circ s_3).$$
- An operation $\circ$ on S is <u>commutative</u> if for all $s_1, s_2 \in S$
$$s_1 \circ s_2 = s_2 \circ s_1$$

∘ <u>Examples</u> - The maps:

$$\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \quad \text{given by}$$
$$(r, s) \mapsto rs \quad \text{and}$$

$$- : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \quad \text{given by}$$
$$(i, j) \mapsto i - j$$

$\boxed{4}$ (cont'd.)

are operations on $\mathbb{R}$ and $\mathbb{Z}$, respectively. The operation $\cdot$ is associative and commutative, whereas $-$ is neither.

$\boxed{5}$

- The positive integers is the set $\mathbb{Z}_{>0} = \{1, 2, 3, \ldots\}$. They are great for counting things.
- The nonnegative integers is the set
$$\mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \ldots\}.$$
With this set, we can add!
- The integers is the set
$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$
With $\mathbb{Z}$ in hand, we can also subtract. and even multiply. But wait; in order to divide we need:
- The rational numbers is the set
$$\mathbb{Q} = \left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z},\ b \neq 0 \right\}.$$ Yea!
- The real numbers is the set
$$\mathbb{R} = \{\text{all decimal expansions}\}$$

$\boxed{5}$ (cont'd.)

This set allows us to solve certain polynomials, eg. $x^2-2=0$.

So, now we can find $\sqrt{2}$.

But what about solving $x^2+1=0$?

- The complex numbers is the set

$$\mathbb{C}=\{a+bi \mid a,b\in\mathbb{R} \text{ and } i=\sqrt{-1}\}$$

With $\mathbb{C}$ we can find $\sqrt{-15920}$, etc.

$\boxed{6}$ Show that the empty set is a subset of every set.

[proof] Proceed with proof by contradiction.

Assume: There exists a set S such that $\phi\not\subseteq S$.

Then there is a $x\in\phi$ such that $x\notin S$.

Contradiction since ϕ has no elements.

Thus, $\phi\subseteq S$.

$\boxtimes$

$\boxed{10}$ Let S, T, and U be sets and let $f: S \to T$ and $g: T \to U$ be functions.

(a) If f and g are injective, $g \circ f$ is injective.

[proof]

To show: If f and g are injective, then $g \circ f$ is injective

Assume: f and g are injective.

To show: $g \circ f$ is injective.

Assume: $(g \circ f)(s_1) = (g \circ f)(s_2)$ for $s_1, s_2 \in S$.

To show: $s_1 = s_2$.

Then $g(f(s_1)) = g(f(s_2))$ for $f(s_1), f(s_2) \in T$.

Then $f(s_1) = f(s_2)$ by injectivity of g, for $s_1, s_2 \in S$.

Then $s_1 = s_2$ by injectivity of f.

Thus, $g \circ f$ is injective. ∎

(b) If f and g are surjective, then $g \circ f$ is surjective.

[proof]

Assume: f and g are surjective.

To show: $g \circ f$ surjective.

Let $u \in U$.

Since g is surjective, there is a $t \in T$ such that $g(t) = u$.

$\boxed{10}$ (cont'd.)

(b) (cont'd.) Then since f is surjective there exists an $s \in S$ such that $f(s) = t$.

Then $g(f(s)) = u$.

Then $(g \circ f)(s) = u$.

Thus, $g \circ f$ is surjective. ∎

(c) If f and g are bijective, then $g \circ f$ is bijective.

[proof]

Assume: f and g are bijective

To show: (ca) $g \circ f$ is injective.

(cb) $g \circ f$ is surjective.

(ca) Then $g \circ f$ is injective by (a).

(cb) Then $g \circ f$ is surjective by (b).

Thus, $g \circ f$ is bijective. ∎

(11) Assume: $f: S \to T$ is a function
To Show: $F = \{f^{-1}(t) \mid t \in T\}$ is a partition of S.
That is,
(a) If $s \in S$, then there exists $t \in T$ with $s \in f^{-1}(t)$.
(b) If $t_1, t_2 \in T$ with $f^{-1}(t_1) \cap f^{-1}(t_2) \neq \emptyset$, then $f^{-1}(t_1) = f^{-1}(t_2)$.

(a) Assume: $s \in S$
To Show: There exists $t \in T$ with $s \in f^{-1}(t)$.
Since $s \in S$, we may define $t = f(s) \in T$.
Then, $s \in f^{-1}(t)$ by definition of a fiber.
So, if $s \in S$, then there exists $t \in T$ with $s \in f^{-1}(t)$.
(b) Assume: $t_1, t_2 \in T$ with $f^{-1}(t_1) \cap f^{-1}(t_2) \neq \emptyset$
To Show: $f^{-1}(t_1) = f^{-1}(t_2)$.
Let $s \in f^{-1}(t_1) \cap f^{-1}(t_2)$. Then $s \in f^{-1}(t_1)$, so $f(s) = t_1$. Also, $s \in f^{-1}(t_2)$, so $f(s) = t_2$.
Then $t_1 = t_2$, so $f^{-1}(t_1) = f^{-1}(t_2)$.
So, if $f^{-1}(t_1) \cap f^{-1}(t_2) \neq \emptyset$, then $f^{-1}(t_1) = f^{-1}(t_2)$.
So, if $f: S \to T$ is a function, then $F = \{f^{-1}(t) \mid t \in T\}$ is a partition of S.

(12a) • First we will show that f' is well-defined.
Assume: $f: S \to T$ is a function
To Show: $f': S \to \text{im}(f)$ is well-defined.
That is,
(a) For every $s \in S$, $f'(s) \in \text{im} f$.
(b) If $s_1 = s_2$ then $f(s_1) = f(s_2)$

(a) Assume: $s \in S$
To Show: $f'(s) \in \text{im}(f)$.
Recall that $f: S \to T$ is a function, and thus $f(s) \in T$ is an element of $\text{im}(f)$ by definition of $\text{im}(f)$. So,
$$f'(s) = f(s) \in \text{im}(f),$$
so if $s \in S$, then $f'(s) \in \text{im}(f)$.

12a con't

(b) Assume: $s_1, s_2 \in S$ with $s_1 = s_2$
To Show: $f'(s_1) = f'(s_2)$.
Recall that $f: S \to T$ is a function, so $f(s_1) = f(s_2)$. Also,
$$f'(s_1) = f(s_1)$$
and
$$f'(s_2) = f(s_2),$$
so $f'(s_1) = f'(s_2)$.
So, if $s_1 = s_2$, then $f'(s_1) = f'(s_2)$.
So, $f': S \to im(f)$ is well-defined.

- Now, we will show f' is surjective:
Assume: $f: S \to T$ is a function.
To Show: $f': S \to im(f)$ is surjective.
That is, if $t \in im(f)$, then there exists $s \in S$ with $f'(s) = t$.
Assume: $t \in im(f)$.
To Show: There exists $s \in S$ with $f'(s) = t$.
Since $t \in im(f)$, there must exist $s \in S$ with $f(s) = t$. Then $f'(s) = f(s) = t$.
So, if $f: S \to T$ is a function, then $f': S \to im(f)$ is surjective.

12b
- First we will show $\hat{f}'$ is well-defined.
Assume: $f: S \to T$ is a function.
To Show: $\hat{f}: F \to T$ is well-defined.
That is,
(a) If $t \in T$, then $\hat{f}(f^{-1}(t)) \in T$
(b) If $t_1, t_2 \in T$ with $f^{-1}(t_1) = f^{-1}(t_2)$, then $\hat{f}(f^{-1}(t_1)) = \hat{f}(f^{-1}(t_2))$.

(a) Assume: $t \in T$
To Show: $\hat{f}(f^{-1}(t)) \in T$
Note that $\hat{f}(f^{-1}(t)) = t$, which is an element of T.
So if $t \in T$, then $\hat{f}(f^{-1}(t)) \in T$.

12b con't

(b) Assume: $t_1, t_2 \in T$ with $f^{-1}(t_1) = f^{-1}(t_2)$.
To Show: $\hat{f}(f^{-1}(t_1)) = \hat{f}(f^{-1}(t_2))$.
Let $s \in f^{-1}(t_1)$. Then $f(s) = t_1$.
Since $f^{-1}(t_1) = f^{-1}(t_2)$, then $s \in f^{-1}(t_2)$.
Then $f(s) = t_2$. Thus $t_1 = t_2$.
Finally,
$$\hat{f}(f^{-1}(t_1)) = t_1$$
and
$$\hat{f}(f^{-1}(t_2)) = t_2,$$
so $\hat{f}(f^{-1}(t_1)) = \hat{f}(f^{-1}(t_2))$.
So if $f^{-1}(t_1) = f^{-1}(t_2)$, then $\hat{f}(f^{-1}(t_1)) = \hat{f}(f^{-1}(t_2))$.
So, if $f: S \to T$ is a function, then $\hat{f}$ is well-defined

- Now we will show that $\hat{f}$ is injective.

Assume: $f: S \to T$ is a function
To Show: $\hat{f}: F \to T$ is injective.
That is, if $f^{-1}(t_1), f^{-1}(t_2) \in F$ with $\hat{f}(f^{-1}(t_1)) = \hat{f}(f^{-1}(t_2))$, then $f^{-1}(t_1) = f^{-1}(t_2)$.
Assume: $\hat{f}(f^{-1}(t_1)) = \hat{f}(f^{-1}(t_2))$
To Show: $f^{-1}(t_1) = f^{-1}(t_2)$.
Note that
$$\hat{f}(f^{-1}(t_1)) = t_1$$
and
$$\hat{f}(f^{-1}(t_2)) = t_2,$$
so $t_1 = t_2$. Thus $f^{-1}(t_1) = f^{-1}(t_2)$, as desired.
So, if $f: S \to T$ is a function, then $\hat{f}$ is injective.

12c
- First we will show that $\hat{f}'$ is well-defined.

Assume: $f: S \to T$ is a function
To Show: $\hat{f}'$ is well-defined.
Since $f: S \to T$ is a function, $\hat{f}: F \to T$ is a well-defined function by problem 12(b).
Then, since $\hat{f}$ is well-defined, $\hat{f}'$ is well-defined by problem 12(c).
So if $f: S \to T$ is a function, then $\hat{f}'$ is well-defined.

12C(con't). Next we will show $\hat{f}'$ is bijective.

Assume: $f: S \to T$ is a function.

To Show: $\hat{f}'$ is bijective.

That is,

(a) $\hat{f}'$ is surjective, and

(b) $\hat{f}'$ is injective.

(a) To Show: $\hat{f}'$ is surjective

Since $f: S \to T$ is a function, we know that $\hat{f}: F \to T$ is a function by 12(b). Thus, $\hat{f}'$ is a surjective function by 12(c).

(b) To Show: $\hat{f}'$ is injective.

Assume: $f^{-1}(t_1) = f^{-1}(t_2)$

To Show: $\hat{f}'(f^{-1}(t_1)) = \hat{f}'(f^{-1}(t_2))$.

Recall that $\hat{f}$ is injective, so $\hat{f}(f^{-1}(t_1)) = \hat{f}(f^{-1}(t_2))$. Since $\hat{f}'(f^{-1}(t)) = \hat{f}(f^{-1}(t))$ for all $f^{-1}(t) \in F$, we know that $\hat{f}'(f^{-1}(t_1)) = \hat{f}'(f^{-1}(t_2))$.

So, $\hat{f}'$ is injective.

So, if $f: S \to T$ is a function, then $\hat{f}'$ is bijective.

(13) To Show: $\Psi: 2^S \to \{0,1\}^S$ is a bijection.

That is,

(a) Ψ is surjective, and

(b) Ψ is injective

(a) To Show: Ψ is surjective

Assume: $f \in \{0,1\}^S$

To Show: There exists $T \in 2^S$ such that $\Psi: T \to f$.

Given $f \in \{0,1\}^S$, define

$T = \{s \in S \mid f(s) = 1\} \subseteq S$

Then $f(s) = 1$ for all $s \in T$, and $f(s) = 0$ for all $s \notin T$. Thus $f = f_T$. So, $\Psi(T) = f_T = f$.

So, Ψ is surjective.

13 Con't

(b) To Show: Ψ is injective
Assume: $T_1, T_2 \in 2^S$ with $\Psi(T_1) = \Psi(T_2)$
To Show: $T_1 = T_2$.
That is,
(ba) $T_1 \subseteq T_2$, and
(bb) $T_2 \subseteq T_1$.

(ba) To Show: $T_1 \subseteq T_2$.
Assume: $t \in T_1$
To Show: $t \in T_2$.
Note that $t \in T_1$, so $f_{T_1}(t) = 1$.
Since $f_{T_1} = \Psi(T_1) = \Psi(T_2) = f_{T_2}$,
it follows that $f_{T_2}(t) = 1$, so
$t \in T_2$.
So, $T_1 \subseteq T_2$
(bb) To Show: $T_2 \subseteq T_1$
Assume: $t \in T_2$
To Show: $t \in T_1$.
Note that $t \in T_2$, so $f_{T_2}(t) = 1$.
Since $f_{T_2} = \Psi(T_2) = \Psi(T_1) = f_{T_1}$,
it follows that $f_{T_1}(t) = 1$, so
$t \in T_1$.
So, $T_2 \subseteq T_1$
So, $T_1 = T_2$
So, Ψ is injective.
So, Ψ is bijective.

$\boxed{14}$ Let $\circ : S \times S \to S$ be an associative operation on the set S.

(a) Show that if S contains an identity for $\circ$ then it is unique.

[proof]

Assume: e_1 and e_2 are identities for $\circ$.

To show: $e_1 = e_2$.

Then $e_1 \circ e_2 = e_1$ since e_1 is an identity for $\circ$.

Then $e_1 \circ e_2 = e_2$ since e_2 is an identity for $\circ$.

Thus, $e_1 = e_2$. ∎

(b) Let e be an identity for an associative operation $\circ$ on S. Let $s \in S$. Show that if s has an inverse then it is unique.

[proof]

Assume: s_1 and s_2 are inverses for s.

To show: $s_1 = s_2$.

Then $s_1 = (s_2 \circ s) \circ s_1 = s_2 \circ (s \circ s_1) = s_2$ by associativity of $\circ$.

Thus, $s_1 = s_2$. ∎

(16) • A <u>partition</u> of a set S is a collection of subsets S_σ such that:

(a) If $s \in S$, then there exists some S_σ with $s \in S_\sigma$

(b) if $S_\sigma \cap S_\beta \neq \phi$, then $S_\sigma = S_\beta$.

• Let S be a set and let $\sim$ be an equivalence relation on S. The <u>equivalence class</u> of an element $s \in S$ is the set

$$[s] = \{t \in S \mid t \sim s\}$$

• <u>Claim</u>: (a) If $\sim$ is an equivalence relation on a set S, then the set of equivalence classes of $\sim$ is a partition of S.

(b) If $\{S_\sigma\}$ is a partition of a set S, then the relation defined by

$s \sim t$ if s and t are in the same S_σ

is an equivalence relation on S.

(a) Assume: $\sim$ is an equivalence relation on S

To Show: $\{[a] \mid a \in S\}$ is a partition of S.

That is,

(aa) If $s \in S$, then there exist some $[a]$ with $s \in [a]$.

(ab) If $[a] \cap [b] \neq \phi$, then $[a] = [b]$.

(aa) Assume: $s \in S$.

To Show: there exists $[a] \in \{[a] : a \in S\}$ with $s \in [a]$.

Note that $s \in S$, so if we let $a = s$, then $s \sim a$ because $\sim$ is reflexive. Thus $s \in [a]$.

(ab) Assume: $a, b \in S$ with $[a] \cap [b] \neq \phi$.

To Show: $[a] = [b]$.

That is,

(aba) $[a] \subseteq [b]$

(abb) $[b] \subseteq [a]$.

Note that $[a] \cap [b] \neq \phi$, so there exists some $c \in S$ with $c \in [a] \cap [b]$.

(aba) To Show: $[a] \subseteq [b]$

Assume: $d \in [a]$,

To Show: $d \in [b]$.

16 Con't

Note that $d \in [a]$, so $d \sim a$.
But $a \sim c$, so $d \sim c$ by transitivity.
Since $c \sim b$, then $d \sim b$, so $d \in [b]$.
So $[a] \subseteq [b]$

(abb) To Show: $[b] \subseteq [a]$.
Assume: $d \in [b]$
To Show: $d \in [a]$
Note that $d \in [b]$, so $d \sim b$.
But $c \sim b$, so $d \sim c$ by transitivity.
Since $c \sim a$, then $d \sim a$, so $d \in [a]$
So $[b] \subseteq [a]$.

So, $[a] = [b]$

(b) Assume: $\{S_\sigma\}$ is a partition of a set S
To Show: $\sim$ is an equivalence relation on S.
That is,
(ba) If $a \in S$, then $a \sim a$
(bb) If $a, b \in S$ with $a \sim b$, then $b \sim a$
(bc) if $a, b, c \in S$ with $a \sim b$, $b \sim c$, then $a \sim c$

(ba) Assume: $a \in S$
To Show: $a \sim a$.
Note that $a \in S$, so $a \in S_\beta$ for some S_β.
Then $a, a \in S_\beta$, so $a \sim a$
So if $a \in S$, then $a \sim a$.

(bb) Assume: $a, b \in S$ with $a \sim b$
To Show: $b \sim a$.
Since $a \sim b$, a and b are in the same S_β.
Thus b and a are in the same S_β, so $b \sim a$
So if $b \sim a$, then $a \sim b$.

(bc) Assume: $a, b, c \in S$ $a \sim b$ and $b \sim c$
To Show: $a \sim c$.
Since $a \sim b$, a and b are in the same S_σ
Since $b \sim c$, b and c are in the same S_β.
Since $b \in S_\sigma \cap S_\beta$, then $S_\sigma = S_\beta$. Thus a and c are in the same S_σ, so $a \sim c$

So, $\sim$ is an equivalence relation on S.

HW 1 Problem 7 Let A B and C be sets. Show that

(a) $(A \cup B) \cup C = A \cup (B \cup C)$

(b) $A \cup B = B \cup A$

(c) $A \cup \phi = A$

(d) $(A \cap B) \cap C = A \cap (B \cap C)$

(e) $A \cap B = B \cap A$

(f) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof

(a) To show: (aa) $(A \cup B) \cup C \subseteq A \cup (B \cup C)$

(ab) $A \cup (B \cup C) \subseteq (A \cup B) \cup C$

(aa) To show: If $x \in (A \cup B) \cup C$ then $x \in A \cup (B \cup C)$

Assume $x \in (A \cup B) \cup C$.

To show: $x \in A \cup (B \cup C)$.

We know $x \in A \cup B$ or $x \in C$.

So $x \in A$ or $x \in B$ or $x \in C$.

So $x \in A$ or $x \in B \cup C$.

So $x \in A \cup (B \cup C)$.

(ab) To show: If $x \in A \cup (B \cup C)$ then $x \in (A \cup B) \cup C$.

Assume $x \in A \cup (B \cup C)$

To show: $x \in (A \cup B) \cup C$

We know $x \in A$ or $x \in B \cup C$

So $x \in A$ or $x \in B$ or $x \in C$.

So $x \in A \cup B$ or $x \in C$.

So $x \in (A \cup B) \cup C$.

So $(A \cup B) \cup C \subseteq A \cup (B \cup C)$ and $A \cup (B \cup C) \subseteq (A \cup B) \cup C$.

So $(A \cup B) \cup C = A \cup (B \cup C)$.

(b) To show: $A \cup B = B \cup A$

To show: (ba) $A \cup B \subseteq B \cup A$

(bb) $B \cup A \subseteq A \cup B$

(ba) To show: If $x \in A \cup B$ then $x \in B \cup A$

Assume $x \in A \cup B$

Then $x \in A$ or $x \in B$.

So $x \in B$ or $x \in A$.

So $x \in B \cup A$. So $A \cup B \subseteq B \cup A$

(bb) To show: If ~~$x \in A \cup B$ then~~ $x \in B \cup A$ then $x \in A \cup B$

Assume $x \in B \cup A$

So $x \in B$ or $x \in A$. So $x \in A$ or $x \in B$

So $x \in A \cup B$.

So $B \cup A \subseteq A \cup B$.

So $A \cup B = B \cup A$.

(c) To show: $A \cup \phi = A$.

To show: (ca) $A \cup \phi \subseteq A$

(cb) $A \subseteq A \cup \phi$.

(ca) To show: If $x \in A \cup \phi$ then $x \in A$.

Assume $x \in A \cup \phi$.

So $x \in A$ or $x \in \phi$.

So ϕ has no elements $x \in A$.

So $x \in A$.

So $A \cup \phi \subseteq A$.

(cb) To show: If $x \in A$ then $x \in A \cup \phi$.

Assume $x \in A$.

Then $x \in A$ or $x \in \phi$.

So $x \in A \cup \phi$. So $A \subseteq A \cup \phi$

So $A \cup \phi = A$.

(d) To show: (da) $(A \cap B) \cap C \subseteq A \cap (B \cap C)$

(db) $A \cap (B \cap C) \subseteq (A \cap B) \cap C$

(da) To show: If $x \in (A \cap B) \cap C$ then $x \in A \cap (B \cap C)$.

Assume $x \in (A \cap B) \cap C$.

Then $x \in A \cap B$ and $x \in C$.

So $x \in A$ and $x \in B$ and $x \in C$.

So $x \in A$ and $x \in B \cap C$.

So $x \in A \cap (B \cap C)$.

So $(A \cap B) \cap C \subseteq A \cap (B \cap C)$.

(db) To show: If $x \in A \cap (B \cap C)$ then $x \in (A \cap B) \cap C$.

Assume $x \in A \cap (B \cap C)$.

Then $x \in A$ and $x \in B \cap C$

So $x \in A$ and $x \in B$ and $x \in C$.

So $x \in A \cap B$ and $x \in C$.

So $x \in (A \cap B) \cap C$.

So $A \cap (B \cap C) \subseteq (A \cap B) \cap C$.

So $A \cap (B \cap C) = (A \cap B) \cap C$.

(e) To show: $A \cap B = B \cap A$

To show: (ea) $A \cap B \subseteq B \cap A$

(eb) $B \cap A \subseteq A \cap B$.

(ea) To show: If $x \in A \cap B$ then $x \in B \cap A$.

Assume $x \in A \cap B$.

Then $x \in A$ and $x \in B$.

So $x \in B$ and $x \in A$.

So $x \in B \cap A$

So $A \cap B \subseteq B \cap A$.

(eb) To show: If $x \in B \cap A$ then $x \in A \cap B$

Assume $x \in B \cap A$

Then $x \in B$ and $x \in A$

So $x \in A$ and $x \in B$.

So $x \in A \cap B$.

So $B \cap A \subseteq A \cap B$.

So $A \cap B = B \cap A$.

(f) To show: (fa) $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

(fb) $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

(fa) ~~$A \cap (B \cup C)$~~

To show: If $x \in A \cap (B \cup C)$ then $x \in (A \cap B) \cup (A \cap C)$

Assume $x \in A \cap (B \cup C)$

Then $x \in A$ and $x \in B \cup C$.

So $x \in A$ and, $x \in B$ or $x \in C$.

So, either $x \in A$ and $x \in B$, or $x \in A$ and $x \in C$.

So $x \in A \cap B$ or $x \in A \cap C$.

So $x \in (A \cap B) \cup (A \cap C)$

So $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

(6)

(fb) To show: If $x \in (A \cap B) \cup (A \cap C)$ then $x \in A \cap (B \cup C)$.

Assume $x \in (A \cap B) \cup (A \cap C)$.

So $x \in A \cap B$ or $x \in (A \cap C)$.

So $x \in A$ and $x \in B$, or $x \in A$ and $x \in C$.

So $x \in A$ and either $x \in B$ or $x \in C$.

So $x \in A \cap (B \cup C)$.

So $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

So $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

HW1 Problem 15

Let S and T be sets and let ι_S and ι_T be the identity maps on S and T, respectively.

(a) Show that for any function $f: S \to T$,

$$\iota_T \circ f = f \quad \text{and} \quad f \circ \iota_S = f.$$

Proof Assume $f: S \to T$ is a function

To show (aa) $\iota_T \circ f = f$

(ab) $f \circ \iota_S = f$.

(aa) To show: If $s \in S$ then $(\iota_T \circ f)(s) = f(s)$.

Assume $s \in S$.

To show: $(\iota_T \circ f)(s) = f(s)$.

$(\iota_T \circ f)(s) = \iota_T(f(s)) = f(s)$, since ι_T is the identity function on T.

So $\iota_T \circ f = f$.

(ab) To show: If $s \in S$ then $(f \circ \iota_S)(s) = f(s)$.

Assume $s \in S$.

To show: $(f \circ \iota_S)(s) = f(s)$.

$(f \circ \iota_S)(s) = f(\iota_S(s)) = f(s)$, since ι_S is the identity function on S.

(b) Let $f: S \to T$ be a function. Show that if an inverse function to f exists then it is unique.

<u>Proof</u> Assume $f: S \to T$ is a function and an inverse function to f exists.

To show: If $g_1: T \to S$ and $g_2: T \to S$ are both inverse functions to f then $g_1 = g_2$.

Assume $g_1: T \to S$ and $g_2: T \to S$ are inverse functions to f. So

$$g_1 \circ f = \iota_S, \quad f \circ g_1 = \iota_T, \quad g_2 \circ f = \iota_S \text{ and } f \circ g_2 = \iota_T.$$

To show: $g_1 = g_2$.

By part (a),

$$g_1 = g_1 \circ \iota_T = g_1 \circ (f \circ g_2) = (g_1 \circ f) \circ g_2 = \iota_S \circ g_2 = g_2.$$

So $g_1 = g_2$. //

<u>HW1 Problem 8</u> Let A, B, C be sets. Show that

(1) $A \cup A = A = A \cap A$

(2) $A \cup B = B \cup A$ and $A \cap B = B \cap A$

(3) $(A \cup B) \cup C = A \cup (B \cup C)$ and
$(A \cap B) \cap C = A \cap (B \cap C)$

(4) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(5) $A \cup (A \cap B) = A = A \cap (A \cup B)$.

<u>Proof</u>

(1) To show: (1a) $A \cup A = A$
(1b) $A = A \cap A$.

(1a) $A \cup A = \{x \mid x \in A \text{ or } x \in A\}$
$= \{x \mid x \in A\} = A$

(1b) $A \cap A = \{x \mid x \in A \text{ and } x \in A\}$
$= \{x \mid x \in A\} = A$.

(2) To show: (2a) $A \cup B = B \cup A$
(2b) $A \cap B = B \cap A$

(2a) $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
$= \{x \mid x \in B \text{ or } x \in A\} = B \cup A$.

(2b) $A\cap B = \{x \mid x\in A$ and $x\in B\}$

$= \{x \mid x\in B$ and $x\in A\} = B\cap A.$

(3) To show: (3a) $(A\cup B)\cup C = A\cup(B\cup C)$

(3b) $(A\cap B)\cap C = A\cap(B\cap C)$

(3a) $(A\cup B)\cup C = \{x \mid x\in A\cup B$ or $x\in C\}$

$= \{x \mid x\in A$ or $x\in B$ or $x\in C\}$

$= \{x \mid x\in A$ or $x\in B\cup C\} = A\cup(B\cup C)$

(3b) $(A\cap B)\cap C = \{x \mid x\in A\cap B$ and $x\in C\}$

$= \{x \mid x\in A$ and $x\in B$ and $x\in C\}$

$= \{x \mid x\in A$ and $x\in B\cap C\} = A\cap(B\cap C)$

(4) To show: (4a) $A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$

(4b) $A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$

(4a) $A\cup(B\cap C) = \{x \mid x\in A$ or $x\in B\cap C\}$

$= \{x \mid x\in A$ or $(x\in B$ and $x\in C)\}$

$= \{x \mid x\in A$ or $x\in B$

(4) To show: (4a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(4b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(4a) To show: (4aa) $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

(4ab) $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

(4aa) To show: If $x \in A \cup (B \cap C)$ then $x \in (A \cup B) \cap (A \cup C)$.

Assume $x \in A \cup (B \cap C)$

Then $x \in A$ or $x \in B \cap C$.

Case 1 $x \in A$.

Then $x \in A \cup B$ and $x \in A \cup C$.

So $x \in (A \cup B) \cap (A \cup C)$

Case 2 $x \notin A$.

Then $x \in B \cap C$. So $x \in B$ and $x \in C$.

So $x \in A \cup B$ and $x \in A \cup C$.

So $x \in (A \cup B) \cap (A \cup C)$

So $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

(4ab) To show: If $x \in (A \cup B) \cap (A \cup C)$ then $x \in A \cup (B \cap C)$.

Assume $x \in (A \cup B) \cap (A \cup C)$

Problem 7. For which values of x is the function $f(x) = \begin{cases} x^3 - x^2 + 2x - 2, & \text{if } x \neq 1, \\ 4, & \text{if } x = 1, \end{cases}$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

Then $x \in A \cup B$ and $x \in A \cup C$.

Case 1 $x \in A$.

Then $x \in A \cup (B \cap C)$.

Case 2 $x \notin A$

Then $x \in B$ and $x \in C$.

So $x \in B \cap C$

So $x \in A \cup (B \cap C)$.

So $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

So $(A \cup B) \cap (A \cup C) = A \cup (B \cap C)$

(5) To show: $A \cup (A \cap B) = A = A \cap (A \cup B)$

To show: (5a) $A \cup (A \cap B) = A$.

(5b) $A = A \cap (A \cup B)$.

(5a) By part 4a,

$$A \cup (A \cap B) = (A \cup A) \cap (A \cup B)$$
$$= A \cap (A \cup B) = A.$$

(5b) By part 4b,

$$A \cap (A \cup B) = (A \cap A) \cup (A \cap B)$$
$$= A \cup (A \cap B) = A$$

//