MATH 221 Lecture 5, September 15, 2000

We now know that the derivative with respect to x

$$f \to \boxed{\frac{d}{dx}} \to \frac{df}{dx}$$

satisfies

(1) $\frac{dx}{dx} = 1$

(2) $\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

(3) $\frac{d(cf)}{dx} = c\frac{df}{dx}$ if c is a constant

(4) $\frac{d(uv)}{dx} = u\frac{dv}{dx} + \frac{du}{dx}v$

(5) $\frac{d1}{dx} = 0$

(6) $\frac{dc}{dx} = 0$, if c is a constant

(7) $\frac{dx^n}{dx} = nx^{n-1}$, if $n = 1, 2, 3, \ldots$

(8) $\frac{dx^n}{dx} = nx^{n-1}$, if $n = 0$

(9) $\frac{dx^{-n}}{dx} = (-n)x^{-n-1}$, if $n = 1, 2, 3 \ldots$

(10) $\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx}$.

Example Find $\frac{dy}{dx}$ when $y = (2x-5)^2$

If $g = 2x-5$ then $y = g^2$.

So $\frac{dy}{dx} = \frac{dy}{dg}\frac{dg}{dx} = \frac{dg^2}{dg}\frac{d(2x-5)}{dx} = 2g\frac{d(2x-5)}{dx}$

$= 2(2x-5)\cdot 2 = 8x-20.$

Example Find $\frac{dy}{dx}$ when $y = (2x-5)^2(3x-4)^3$

$\frac{dy}{dx} = \frac{d(2x-5)^2(3x-4)^3}{dx} = (2x-5)^2\frac{d(3x-4)^3}{dx} + \frac{d(2x-5)^2}{dx}(3x-4)^3$

$= (2x-5)^2\frac{dg^3}{dg}\frac{dg}{dx} + \frac{d(2x-5)^2}{dx}(3x-4)^3$

$= (2x-5)^2 3g^2\frac{d(3x-4)}{dx} + 2(2x-5)\frac{d(2x-5)}{dx}(3x-4)^3$

$= (2x-5)^2 3(3x-4)^2\cdot 3 + 2(2x-5)\cdot 2(3x-4)^3$

$= (2x-5)(3x-4)^2(9(2x-5) + 4(3x-4))$

$= (2x-5)(3x-4)^2(30x-61).$

Example Find $\dfrac{d x^{m/n}}{dx}$.

$$\frac{d(x^{m/n})^n}{dx} = \frac{dg^n}{dg}\frac{dg}{dx} \qquad \text{if } g = x^{m/n}$$

$$= n g^{n-1}\frac{d x^{m/n}}{dx} = n(x^{m/n})^{n-1}\frac{d x^{m/n}}{dx}.$$

On the other hand

$$\frac{d(x^{m/n})^n}{dx} = \frac{d x^m}{dx} = m x^{m-1}.$$

So $\quad m x^{m-1} = n(x^{m/n})^{n-1}\dfrac{d x^{m/n}}{dx}.$

So $\quad \dfrac{m x^{m-1}}{n(x^{m/n})^{n-1}} = \dfrac{d x^{m/n}}{dx}$

So $\quad \dfrac{m x^{m-1}}{n(x^{m/n})^{n}(x^{m/n})^{-1}} = \dfrac{d x^{m/n}}{dx}$

So $\quad \dfrac{d x^{m/n}}{dx} = \dfrac{m x^{m-1}\, x^{m/n}}{n x^m} = \dfrac{m}{n} x^{m/n} x^{-1}$

So $\quad \dfrac{d x^{m/n}}{dx} = \dfrac{m}{n} x^{m/n-1}.$

Example Find $\dfrac{dy}{dx}$ if $y = \dfrac{\sqrt{1+x^2}}{\sqrt{1-x^2}}$

$$\frac{dy}{dx} = \frac{d\,\frac{\sqrt{1+x^2}}{\sqrt{1-x^2}}}{dx} = \frac{d\,\frac{(1+x^2)^{\frac12}}{(1-x^2)^{\frac12}}}{dx} = \frac{d\left(\frac{1+x^2}{1-x^2}\right)^{\frac12}}{dx}$$

$$= \frac12\left(\frac{1+x^2}{1-x^2}\right)^{\frac12-1}\frac{d\left(\frac{1+x^2}{1-x^2}\right)}{dx} = \frac12\left(\frac{1+x^2}{1-x^2}\right)^{-\frac12}\frac{d\,(1+x^2)(1-x^2)^{-1}}{dx}$$

$$= \frac12\left(\frac{1-x^2}{1+x^2}\right)^{\frac12}\left((1+x^2)\frac{d(1-x^2)^{-1}}{dx} + \frac{d(1+x^2)}{dx}(1-x^2)^{-1}\right)$$

$$= \frac12\left(\frac{1-x^2}{1+x^2}\right)^{\frac12}\left((1+x^2)(-1)(1-x^2)^{-2}\frac{d(1-x^2)}{dx} + 2x(1-x^2)^{-1}\right)$$

$$= \frac12\left(\frac{1-x^2}{1+x^2}\right)^{\frac12}\left(\frac{(-1)(1+x^2)(-2x)}{(1-x^2)^2} + \frac{2x}{1-x^2}\right)$$

$$= \frac12\left(\frac{1-x^2}{1+x^2}\right)^{\frac12}\left(\frac{2x(1+x^2)}{(1-x^2)^2} + \frac{2x(1-x^2)}{(1-x^2)^2}\right)$$

$$= \frac12\left(\frac{1-x^2}{1+x^2}\right)^{\frac12}\left(\frac{2x(1+x^2+1-x^2)}{(1-x^2)^2}\right)$$

$$= \frac12\,\frac{(1-x^2)^{\frac12}}{(1+x^2)^{\frac12}}\,\frac{4x}{(1-x^2)^2} = \frac{2x}{(1+x^2)^{\frac12}(1-x^2)^{3/2}}.$$

Example Differentiate $\frac{x^2}{1+x^2}$ with respect to x^2.

This is the same problem as:

Find $\frac{dz}{dp}$ when $z = \frac{x^2}{1+x^2}$ and $p = x^2$

Now $\frac{dz}{dx} = \frac{dz}{dp}\frac{dp}{dx}$. So $\frac{dz}{dp} = \frac{dz/dx}{dp/dx}$.

So
$$\frac{dz}{dp} = \frac{\frac{d}{dx}\left(\frac{x^2}{1+x^2}\right)}{\frac{d}{dx}(x^2)} = \frac{\frac{d\, x^2(1+x^2)^{-1}}{dx}}{\frac{dx^2}{dx}} = \frac{\frac{d\, x^2(1+x^2)^{-1}}{dx}}{2x}$$

$$= \frac{x^2 \frac{d(1+x^2)^{-1}}{dx} + \frac{dx^2}{dx}(1+x^2)^{-1}}{2x} = \frac{x^2(-1)(1+x^2)^{-2}\frac{d(1+x^2)}{dx} + 2x(1+x^2)^{-1}}{2x}$$

$$= \frac{\frac{-x^2}{(1+x^2)^2}2x + \frac{2x}{1+x^2}}{2x} = \frac{-x^2}{(1+x^2)^2} + \frac{1}{1+x^2}$$

$$= \frac{-x^2 + 1+x^2}{(1+x^2)^2} = \frac{1}{(1+x^2)^2}$$

Example Find $\frac{dy}{dx}$ when $x^4+y^4 = 4a^2x^2y^2$.

$$\frac{d(x^4+y^4)}{dx} = \frac{d\, 4a^2x^2y^2}{dx}$$

So
$$\frac{dx^4}{dx} + \frac{dy^4}{dx} = 4a^2\frac{dx^2y^2}{dx}$$

So
$$4x^3 + 4y^3\frac{dy}{dx} = 4a^2\left(x^2\frac{dy^2}{dx} + \frac{dx^2}{dx}y^2\right)$$

So
$$4x^3 + 4y^3\frac{dy}{dx} = 4a^2\left(x^2 2y\frac{dy}{dx} + 2xy^2\right)$$
$$= 4a^2x^2 2y\frac{dy}{dx} + 4a^2 2xy^2.$$

So
$$4x^3 - 4a^2 2xy^2 = 4a^2x^2 2y\frac{dy}{dx} - 4y^3\frac{dy}{dx}$$

So
$$4x^3 - 4a^2 2xy^2 = (4a^2x^2 2y - 4y^3)\frac{dy}{dx}$$

So
$$\frac{4x^3 - 4a^2 2xy^2}{4a^2x^2 2y - 4y^3} = \frac{dy}{dx}$$

So
$$\frac{dy}{dx} = \frac{x^3 - 2a^2xy^2}{2a^2x^2y - y^3}.$$

All we did is take the derivative of both sides and then solve for $\frac{dy}{dx}$.

Example Find $\left.\frac{dy}{dx}\right|_{x=3}$ when $y=(x+1)(x+2)$.

$\left.\frac{dy}{dx}\right|_{x=3}$ means: find $\frac{dy}{dx}$ and then plug in $x=3$.

$$\left.\frac{dy}{dx}\right|_{x=3} = \left.\frac{d\,(x+1)(x+2)}{dx}\right|_{x=3}$$

$$= \left.\left((x+1)\frac{d(x+2)}{dx} + \frac{d(x+1)}{dx}(x+2)\right)\right|_{x=3}$$

$$= \left.\big((x+1)+(x+2)\big)\right|_{x=3} = \left.(2x+3)\right|_{x=3}$$

$$= 2\cdot 3+3 = 9.$$

Example Find $\frac{dy}{dx}$ when $x=\frac{3at}{1+t^3}$ and $y=\frac{3at^2}{1+t^3}$.

$$\frac{dy}{dx} = \frac{d(xt)}{dx} \qquad \text{since } y=\frac{3at^2}{1+t^3} = \left(\frac{3at}{1+t^3}\right)t = xt.$$

So $$\frac{dy}{dx} = x\frac{dt}{dx} + \frac{dx}{dx}\cdot t = x\frac{dt}{dx}+t.$$

What is $\frac{dt}{dx}$?? Since $\frac{dx}{dx} = \frac{dx}{dt}\frac{dt}{dx}$, $\frac{dt}{dx} = \frac{dx/dx}{dx/dt} = \frac{1}{dx/dt}$

So
$$\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}} = \frac{1}{\frac{d\left(\frac{3at}{1+t^3}\right)}{dt}} = \frac{1}{\frac{d\,(3at)(1+t^3)^{-1}}{dt}}$$

$$= \frac{1}{3at(-1)(1+t^3)^{-2}\cdot\frac{d(1+t^3)}{dt} + 3a(1+t^3)^{-1}}$$

$$= \frac{1}{\frac{-3at}{(1+t^3)^2}\cdot 3t^2 + \frac{3a}{1+t^3}}$$

$$= \frac{1}{\frac{-9at^3+3a(1+t^3)}{(1+t^3)^2}} = \frac{(1+t^3)^2}{-9at^3+3at^3+3a}$$

$$= \frac{(1+t^3)^2}{3a-6at^3}$$

So
$$\frac{dy}{dx} = x\frac{dt}{dx}+t = \frac{3at}{1+t^3}\,\frac{(1+t^3)^2}{3a(1-2t^3)} + t$$

$$= \frac{t(1+t^3)}{1-2t^3} + \frac{t(1-2t^3)}{1-2t^3} = \frac{t+t^4+t-2t^4}{1-2t^3}$$

$$= \frac{2t-t^4}{1-2t^3}.$$