MATH 221 Lecture 33, November 29, 2000 ①

Example Find the area of the surface obtained by rotating the curve determined by $x = a\cos^3\theta$, $y = a\sin^3\theta$ about the x-axis.

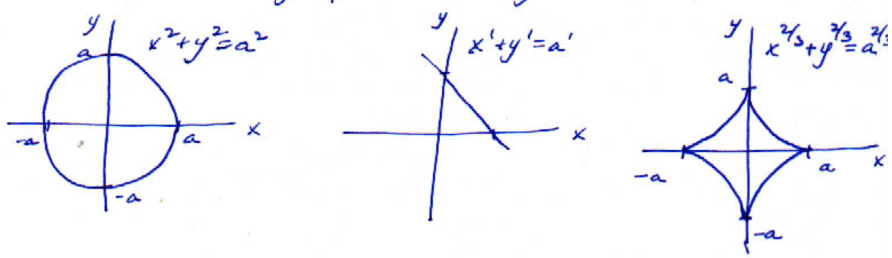
To graph this:

$\cos^3\theta = \frac{x}{a}$ and $\sin^3\theta = \frac{y}{a}$.

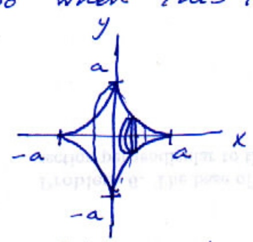
So $\cos\theta = \left(\frac{x}{a}\right)^{1/3}$ and $\sin\theta = \left(\frac{y}{a}\right)^{1/3}$

Since $\cos^2\theta + \sin^2\theta = 1$, $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{a}\right)^{2/3} = 1$

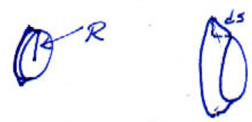
So we have to graph $x^{2/3} + y^{2/3} = a^{2/3}$.



So when this is rotated about the x-axis



Slice:



Surface area of a slice: $\pi R^2 ds$.

Add up slices from $x=0$ to $x=a$ and then multiply by 2

②

$$\text{Surface area} = 2\int_{x=0}^{x=a} \pi R^2\, ds$$

$$= 2\int_{x=0}^{x=a} \pi y^2 \sqrt{(dx)^2+(dy)^2}$$

$$= 2\int_{x=0}^{x=a} \pi y^2 \frac{\sqrt{(dx)^2+(dy)^2}}{d\theta}\, d\theta$$

$$= 2\int_{x=0}^{x=a} \pi y^2 \sqrt{\frac{(dx)^2+(dy)^2}{(d\theta)^2}}\, d\theta$$

$$= 2\int_{x=0}^{x=a} \pi y^2 \sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

Since $x = a\cos^3\theta$ and $y = a\sin^3\theta$

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta \quad\text{and}\quad \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

So

$$\text{Surface area} = 2\int_{x=0}^{x=a} \pi y^2 \sqrt{(-3a\cos^2\theta\sin\theta)^2+(3a\sin^2\theta\cos\theta)^2}\, d\theta$$

$$= 2\int_{x=0}^{x=a} \pi a^2\sin^6\theta \sqrt{9a^2\cos^4\theta\sin^2\theta+9a^2\sin^4\theta\cos^2\theta}\, d\theta$$

③

$$= 2\int_{x=0}^{x=a} \pi a^2 \sin^6\theta \sqrt{9a^2\sin^2\theta\cos^2\theta(\cos^2\theta+\sin^2\theta)}\, d\theta$$

$$= 2\int_{x=0}^{x=a} \pi a^2 \sin^6\theta \, 3a\sin\theta\cos\theta\, d\theta$$

$$= 2\int_{x=0}^{x=a} 3\pi a^3 \sin^7\theta\cos\theta\, d\theta = 6\pi a^3 \frac{\sin^8\theta}{8}\Bigg|_{x=0}^{x=a}$$

$$= \frac{6\pi a^3}{8}\sin^8\theta\Bigg|_{a\cos^3\theta=0}^{a\cos^3\theta=a} = \frac{3\pi a^3}{4}\sin^8\theta\Bigg|_{\cos\theta=0}^{\cos\theta=1}$$

$$= \frac{3a^3\pi}{4}\sin^8\theta\Bigg|_{\theta=\frac{\pi}{2}}^{\theta=0} = \frac{3\pi a^3}{4}\sin^8 0 - \frac{3\pi a^3}{4}\sin^8\frac{\pi}{2}$$

$$= -\frac{3\pi a^3}{4}\cdot 1^8 = -\frac{3\pi a^3}{4}.$$

So the surface area is $\frac{3\pi a^3}{4}$.

④

Moments and center of mass

A moment and a center of mass are the same thing.

The center of mass is the average position of the mass in an object.

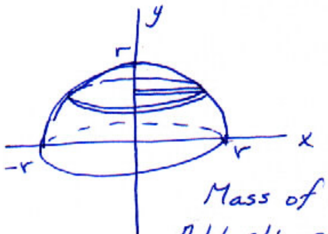
$$\text{Center of mass} = \frac{\text{position of mass}\cdot\text{mass}}{\text{mass}}$$

$$= \frac{\int \begin{pmatrix}\text{position of}\\ \text{a slice}\end{pmatrix}\cdot\begin{pmatrix}\text{mass}\\ \text{of a slice}\end{pmatrix}}{\int \begin{pmatrix}\text{mass of}\\ \text{a slice}\end{pmatrix}}.$$

Note: $\text{mass of a slice} = \begin{pmatrix}\text{volume of}\\ \text{slice}\end{pmatrix}\cdot\begin{pmatrix}\text{density of}\\ \text{the slice}\end{pmatrix}.$

Center of mass and center of gravity are the same thing.

Example: Find the center of mass of a solid hemisphere of radius r if its density at a point P is proportional to the distance between P and the base of the hemisphere.



Slice: 

Volume of a slice: $\pi R^2 \, dy$

density of a slice = height of slice

Mass of a slice = $\pi R^2 \, dy$ (height of slice)

Add slices from $y=0$ to $y=r$

$$\text{Center of mass} = \frac{\int \left(\begin{array}{c}\text{center of}\\ \text{mass of}\\ \text{slice}\end{array}\right)\left(\begin{array}{c}\text{mass of}\\ \text{slice}\end{array}\right)}{\int \left(\begin{array}{c}\text{mass of}\\ \text{slice}\end{array}\right)} = \frac{\int_{y=0}^{y=r} y \, \pi R^2 \, dy \left(\begin{array}{c}\text{height of}\\ \text{slice}\end{array}\right)}{\int_{y=0}^{y=r} \pi R^2 \, dy \left(\begin{array}{c}\text{height of}\\ \text{slice}\end{array}\right)}$$

$$= \frac{\int_{y=0}^{y=r} y \pi x^2 \, dy \cdot y}{\int_{y=0}^{y=r} \pi x^2 \, dy \, y} = \frac{\int_{y=0}^{y=r} \pi x^2 y^2 \, dy}{\int_{y=0}^{y=r} \pi x^2 y \, dy} = \frac{\int_{y=0}^{y=r} \pi (r^2 - y^2) y^2 \, dy}{\int_{y=0}^{y=r} \pi (r^2 - y^2) y \, dy}$$

$$= \frac{\int_{y=0}^{y=r} (\pi r^2 y^2 - \pi y^4) \, dy}{\int_{y=0}^{y=r} (\pi r^2 y - \pi y^3) \, dy} = \frac{\left. \frac{\pi r^2 y^3}{3} - \frac{\pi y^5}{5} \right|_{y=0}^{y=r}}{\left. \frac{\pi r^2 y^2}{2} - \frac{\pi y^4}{4} \right|_{y=0}^{y=r}}$$

$$= \frac{\frac{\pi r^5}{3} - \frac{\pi r^5}{5} - (0-0)}{\frac{\pi r^4}{2} - \frac{\pi r^4}{4} - (0-0)} = \frac{\pi r^5 \left(\frac{1}{3} - \frac{1}{5}\right)}{\pi r^4 \left(\frac{1}{2} - \frac{1}{4}\right)}$$

$$= \frac{r\left(\frac{2}{15}\right)}{\left(\frac{1}{4}\right)} = r \, \frac{2}{15} \cdot 4 = \frac{8r}{15}.$$

So the center of mass is at $\left(0, \frac{8r}{15}\right)$.