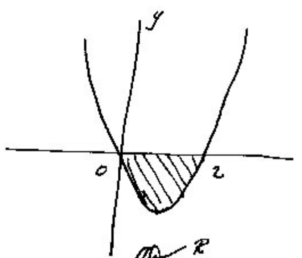
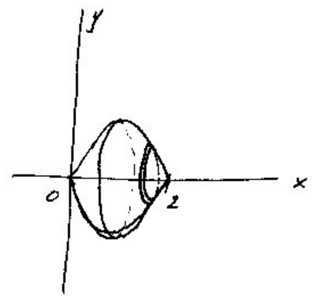
MATH 221 Lecture 28, November 13, 2000 ①

Example Find the volume generated by the area bounded by $y = x^2 - 2x$ and $y = 0$ when it is rotated about the x-axis.




Slice: 


Volume of a slice: $\pi R^2 dx$

Add slices from $x=0$ to $x=2$.

$$\int_{x=0}^{x=2} \pi R^2 dx = \int_{x=0}^{x=2} \pi(-y)^2 dx = \int_{x=0}^{x=2} \pi y^2 dx = \int_{x=0}^{x=2} \pi(x^2-2x)^2 dx$$

$$= \int_{x=0}^{x=2} \pi(x^4 - 4x^3 + 4x^2)\,dx = \pi\left(\frac{x^5}{5} - \frac{4x^4}{4} + \frac{4x^3}{3}\right)\Bigg|_{x=0}^{x=2}$$

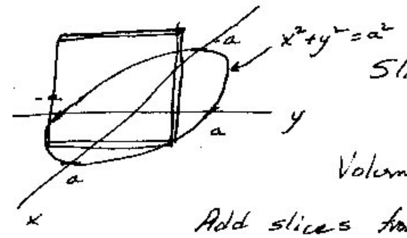
$$= \pi\left(\frac{2^5}{5} - 2^4 + \frac{4}{3}2^3\right) - \pi(0-0+0) = 2^3\pi\left(\frac{2^2}{5} - 2 + \frac{4}{3}\right)$$

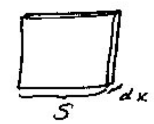
$$= 8\pi\left(\frac{4}{5} - 2 + \frac{4}{3}\right) = 8\pi\left(\frac{-6}{5} + \frac{4}{3}\right) = 8\pi\left(\frac{-18}{15} + \frac{20}{15}\right) = \frac{8\pi \cdot 2}{15}$$

$$= \frac{16\pi}{15}$$

Example The base of a solid is $x^2+y^2=a^2$. ②

Each plane section, perpendicular to the x-axis, is a square, with one edge of the square in the base of the solid. Find the volume.




Slice: 


Volume of slice: $S^2 dx$

Add slices from $x=-a$ to $x=a$.

$$\int_{x=-a}^{x=a} S^2 dx = \int_{x=-a}^{x=a} (2y)^2 dx = \int_{x=-a}^{x=a} 4y^2 dx = \int_{x=-a}^{x=a} 4(a^2-x^2)\,dx$$

$$= 4\left(a^2x - \frac{x^3}{3}\right)\Bigg|_{x=-a}^{x=a} = 4\left(a^2\cdot a - \frac{a^3}{3}\right) - 4\left(a^2(-a) - \frac{(-a)^3}{3}\right)$$

$$= 4\left(a^3 - \frac{1}{3}a^3\right) - 4\left(-a^3 + \frac{1}{3}a^3\right) = 4\cdot\frac{2}{3}a^3 - 4\left(\frac{-2}{3}a^3\right)$$

$$= 4\cdot\frac{4}{3}a^3 = \frac{16a^3}{3}.$$

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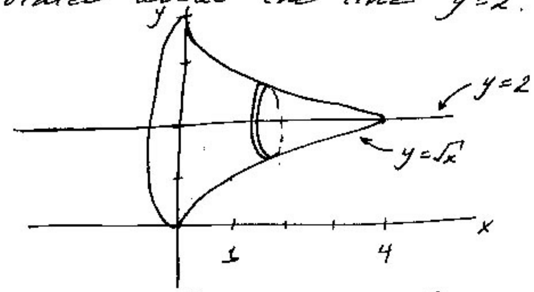
**Example** Find the volume generated when the area bounded by $y=\sqrt{x}$, $y=2$ and $x=0$ is rotated about the line $y=2$.




Slice:  $R$ Volume of a slice: $\pi R^2 dx$

Add slices from $x=0$ to $x=4$.

$$\int_{x=0}^{x=4} \pi R^2 dx = \int_{x=0}^{x=4} \pi (2-y)^2 dx = \int_{x=0}^{x=4} \pi (2-\sqrt{x})^2 dx$$

$$= \int_{x=0}^{x=4} \pi (4 - 4\sqrt{x} + x) dx = \pi\left(4x - \frac{2\cdot 4}{3}x^{3/2} + \frac{x^2}{2}\right)\Bigg|_{x=0}^{x=4}$$

$$= \pi\left(4\cdot 4 - \frac{2}{3}\cdot 4\cdot 4^{3/2} + \frac{4^2}{2}\right) - \pi(0-0+0)$$

$$= \pi\left(16 - \frac{8}{3}\cdot 8 + \frac{16}{2}\right) = 8\pi\left(2 - \frac{8}{3} + 1\right) = 8\pi\cdot\frac{1}{3} = \frac{8\pi}{3}$$

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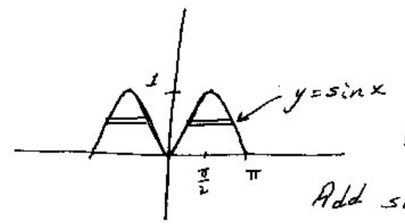
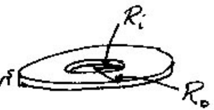
**Example** Find the volume generated when the area ~~generated~~ bounded by $y=\sin x$, $0\le x\le\pi$, and $y=0$ is rotated about the y-axis.




Slice: $dy$ 


Volume of slice: $(\pi R_o^2 - \pi R_i^2)\,dy$

Add slices from $y=0$ to $y=1$.

$$\int_{y=0}^{y=1} (\pi R_o^2 - \pi R_i^2) dy = \int_{y=0}^{y=1} \pi (x_{right}^2 - x_{left}^2) dy$$

$$= \int_{y=0}^{y=1} \pi ((\pi - x_{left})^2 - x_{left}^2) dy = \int_{y=0}^{y=1} \pi(\pi^2 - 2\pi x + x^2 - x^2) dy$$

$$= \int_{y=0}^{y=1} \pi(\pi^2 - 2\pi x)\frac{dy}{dx} dx = \int_{x=0}^{x=\frac{\pi}{2}} \pi(\pi^2 - 2\pi x)\cos x\, dx.$$

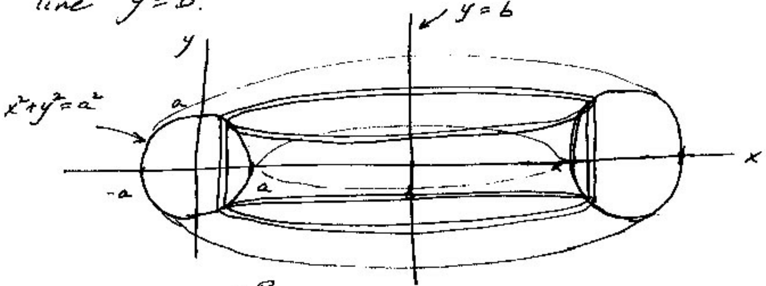
$$= \int_{x=0}^{x=\frac{\pi}{2}} (\pi^3\cos x - 2\pi^2 x\cos x)\, dx$$

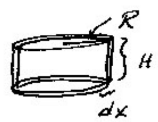
$$= \pi^3 \sin x - 2\pi^2(x\sin x + \cos x)\Big|_{x=0}^{x=\frac{\pi}{2}}$$

$$= \pi^3\sin\frac{\pi}{2} - 2\pi^2\left(\frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{\pi}{2}\right) - (\pi^3\sin 0 - 2\pi^2(0 + \cos 0))$$

$$= \pi^3 - \frac{2\pi^2\cdot\pi}{2} + 2\pi^2 = 2\pi^2.$$

Example Find the volume of a bagel produced by rotating the circle $x^2+y^2=a^2$ about the line $y=b$.




Slice: 


Volume of slice: $2\pi R H\,dx$

Add slices from $x=-a$ to $x=a$.

$$\int_{x=-a}^{x=a} 2\pi R H\,dx = \int_{x=-a}^{x=a} 2\pi(b-x)\,2y\,dx = \int_{x=-a}^{x=a} 2\pi(b-x)\,2\sqrt{a^2-x^2}\,dx$$

$$= \int_{x=-a}^{x=a} \left(4\pi b\sqrt{a^2-x^2} - 4\pi x\sqrt{a^2-x^2}\right)dx$$

$$= \int_{x=-a}^{x=a} 4\pi b\sqrt{a^2-x^2}\,dx - \frac{4\pi}{-2}\int_{x=-a}^{x=a} -2x\sqrt{a^2-x^2}\,dx$$

$$= 4\pi b\left(\begin{array}{c}\text{area of a semicircle}\\ \text{of radius } a\end{array}\right) + 2\pi\left((a^2-x^2)^{3/2}\right)\Big|_{x=-a}^{x=a}$$

$$= 4\pi b\frac{\pi a^2}{2} + 2\pi(0^{3/2}) - 2\pi(0^{3/2}) = \frac{4\pi^2a^2b}{2} = 2\pi^2a^2b.$$