MATH 221 Lecture 23, November 1, 2000 (1)

The chain rule for derivatives:

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}$$

Now

$$f = \int \frac{df}{dx}\,dx = \int \frac{df}{du}\,\frac{du}{dx}\,dx$$

On the other hand

$$f = \int \frac{df}{du}\,du.$$

So

$$\int \frac{df}{du}\,\frac{du}{dx}\,dx = \int \frac{df}{du}\,du$$

So

$$\int (\text{JUNK})\,\frac{du}{dx}\,dx = \int (\text{JUNK})\,du$$

This is THE CHAIN RULE FOR INTEGRALS

Example $\displaystyle\int \frac{4x-5}{2x^2-5x+1}\,dx$ $\qquad u = 2x^2-5x+1$

$$\frac{du}{dx} = 4x-5$$

So

$$\int \frac{4x-5}{2x^2-5x+1}\,dx = \int \frac{1}{u}\,\frac{du}{dx}\,dx = \int \frac{1}{u}\,du$$

$$= \ln u + c = \ln(2x^2-5x+1) + c.$$

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Example $\displaystyle\int \frac{\tan\sqrt{x}\,\sec^2\sqrt{x}}{\sqrt{x}}\,dx$ $\qquad u = \tan\sqrt{x}$

$$\frac{du}{dx} = \sec^2\sqrt{x}\ \frac{1}{2}x^{-\frac{1}{2}}$$

$$= \frac{1}{2}\frac{\sec^2\sqrt{x}}{\sqrt{x}}$$

$$\int 2\tan\sqrt{x}\ \frac{\sec^2\sqrt{x}}{2\sqrt{x}}\,dx$$

$$= \int 2u\,\frac{du}{dx}\,dx = \int 2u\,du = u^2 + c = \tan^2\sqrt{x} + c.$$

Example $\displaystyle\int x\sqrt{3x-2}\,dx = \int \frac{1}{3}\,3x\sqrt{3x-2}\,dx$

$$= \int \frac{1}{3}(3x-2+2)\sqrt{3x-2}\,dx = \int \frac{1}{3}\left((3x-2)^{3/2} + 2(3x-2)^{\frac{1}{2}}\right)dx$$

$$= \frac{1}{3}\left(\frac{2}{5}\frac{(3x-2)^{5/2}}{3} + 2\,\frac{(3x-2)^{3/2}}{3}\,\frac{2}{3}\right) + c$$

$$= \frac{2}{45}(3x-2)^{5/2} + \frac{4}{27}(3x-2)^{3/2} + c$$

Example $\displaystyle\int x\sqrt{x^2-1}\,dx$ $\qquad u = x^2-1$

$$\frac{du}{dx} = 2x$$

$$\int x\sqrt{x^2-1}\,dx = \int \frac{1}{2}\,2x\sqrt{x^2-1}\,dx$$

$$= \int \frac{1}{2}\frac{du}{dx}\sqrt{u}\,dx = \int \frac{1}{2}\sqrt{u}\,du = \int \frac{1}{2}u^{\frac{1}{2}}\,du = \frac{1}{2}\frac{2}{3}u^{3/2} + c$$

$$= \frac{1}{3}u^{3/2} + c = \frac{1}{3}(x^2-1)^{3/2} + c.$$

Example $\int \cos^3 x \, dx = \int \cos x \cos^2 x \, dx$

$$= \int \cos x (1 - \sin^2 x) \, dx = \int (\cos x - \sin^2 x \cos x) \, dx$$

$$= \sin x - \frac{\sin^3 x}{3} + c$$

Example $\int \frac{\ln x^2}{x} \, dx$

$$u = \ln x^2$$

$$\frac{du}{dx} = \frac{1}{x^2} 2x = \frac{2}{x}$$

$$\int \frac{1}{2} \frac{2 \ln x^2}{x} \, dx = \int \frac{1}{2} \frac{2}{x} \ln x^2 \, dx$$

$$= \int \frac{1}{2} \frac{du}{dx} u \, dx = \int \frac{1}{2} u \, du = \frac{1}{2} \frac{u^2}{2} + c = \frac{u^2}{4} + c$$

$$= \frac{(\ln x^2)^2}{4} + c.$$

Example $\int \frac{x}{\sqrt{1+x}} \, dx = \int \frac{x+1-1}{\sqrt{1+x}} \, dx$

$$= \int \left(\frac{x+1}{(1+x)^{\frac{1}{2}}} - \frac{1}{(1+x)^{\frac{1}{2}}} \right) dx = \int \left((x+1)^{\frac{1}{2}} - (x+1)^{-\frac{1}{2}} \right) dx$$

$$= \frac{2}{3}(x+1)^{3/2} - 2(x+1)^{\frac{1}{2}} + c$$

Example $\int x \sqrt{x-1} \, dx = \int (x-1+1) \sqrt{x-1} \, dx$

$$= \int \left((x-1)\sqrt{x-1} + \sqrt{x-1} \right) dx = \int (x-1)^{3/2} + (x-1)^{\frac{1}{2}} \, dx$$

$$= \frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + c$$

Example $\int (1-x) \sqrt{1+x} \, dx = \int \left(-(1+x) + 2 \right) \sqrt{1+x} \, dx$

$$= \int -(1+x)\sqrt{1+x} + 2\sqrt{1+x} \, dx = \int \left(-(1+x)^{3/2} + 2(1+x)^{\frac{1}{2}} \right) dx$$

$$= -\frac{2}{5}(1+x)^{5/2} + 2 \cdot \frac{2}{3}(1+x)^{3/2} + c$$

$$= -\frac{2}{5}(1+x)^{5/2} + \frac{4}{3}(1+x)^{3/2} + c.$$

Example $\int \frac{\sin x}{\sin x - \cos x} \, dx = \int \frac{\sin x - \cos x + \sin x + \cos x}{2(\sin x - \cos x)} \, dx$

$$= \int \frac{\sin x - \cos x}{2(\sin x - \cos x)} + \frac{\sin x + \cos x}{2(\sin x - \cos x)} \, dx$$

$$= \int \frac{1}{2} + \frac{1}{2} \frac{\sin x + \cos x}{\sin x - \cos x} \, dx = \frac{1}{2} x + \frac{1}{2} \ln(\sin x - \cos x) + c.$$