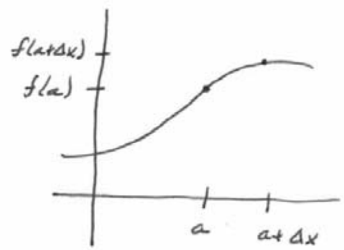
MATH 221, Lecture 20, October 27, 2000 

## Related Rates

IMPORTANT CONCEPT:

The derivative is a rate of change.

$$\left.\frac{df}{dx}\right|_{x=a} = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

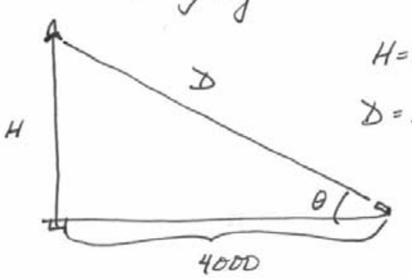



$f(a+\Delta x) - f(a) =$ change in $f$

$\Delta x =$ change in $x$

$\left.\frac{df}{dx}\right|_{x=a}$ measures how $f$ is changing as $x$ changes

Example A TV camera is 4000 feet from the base of a launch pad. A rocket is launched and has a speed of 600 ft/s when it is 3000 ft high. How fast is the distance between the camera and the rocket changing




H = height of rocket

D = distance between camera and rocket.



speed = change in height as time changes $= \frac{dH}{dt}$.

We know $\left.\frac{dH}{dt}\right|_{H=3000} = 600$ ft/s.

We want $\left.\frac{dD}{dt}\right|_{H=3000}$.

From the picture $4000^2 + H^2 = D^2$

So $2H\frac{dH}{dt} = 2D\frac{dD}{dt}$.

So $\frac{dD}{dt} = \frac{2H}{2D}\frac{dH}{dt} = \frac{H}{D}\frac{dH}{dt}$

So

$$\left.\frac{dD}{dt}\right|_{H=3000} = \frac{3000}{\sqrt{3000^2+4000^2}}\left.\frac{dH}{dt}\right|_{H=3000} = \frac{3000}{\sqrt{5000^2}}\cdot 600$$

$$= \frac{3000\cdot 600}{5000} = 3\cdot\frac{600}{5} = 3\cdot 120 = 360 \text{ ft/s}.$$

How fast is the angle of the camera changing?

We want $\left.\frac{d\theta}{dt}\right|_{H=3000}$.

Since

$\tan\theta = \frac{H}{4000}$, $\frac{d\theta}{dt}\sec^2\theta = \frac{1}{4000}\frac{dH}{dt}$

Since $\sec\theta = \frac{1}{\cos\theta} = \frac{D}{4000}$, $\frac{d\theta}{dt}\frac{D^2}{4000^2} = \frac{1}{4000}\frac{dH}{dt}$.

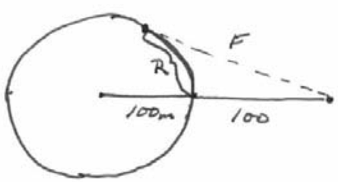
(3)

So $\frac{d\theta}{dt} = \frac{4000^2}{D^2}\,\frac{1}{4000}\,\frac{dH}{dt} = \frac{4000}{D^2}\,\frac{dH}{dt}$

So $\left.\frac{d\theta}{dt}\right|_{H=3000} = \frac{4000}{(3000^2+4000^2)}\cdot\left.\frac{dH}{dt}\right|_{H=3000} = \frac{4000}{5000^2}\cdot 600$

$= \frac{4\cdot 600}{5\cdot 1000} = \frac{4\cdot 120}{1000} = \frac{24}{50}$ radians per sec.

<u>Example</u> A runner runs around a circular track of radius 100m at a speed of 7m/s. The runners friend is standing 200m from the center. How fast is the distance between them changing when their distance is 200m.




speed of runner = change in runner's distance w.r.t. time $= \frac{dR}{dt}$

We want

change in distance between friends w.r.t. time $= \frac{dF}{dt}$

Really we want $\left.\frac{dF}{dt}\right|_{F=200}$.

(4)

$R = 100\cdot\theta$ if $\theta$ is the angle at the point $(x,y)$ at which the runner is at.

$F = \sqrt{(200-x)^2+y^2} = \sqrt{200^2-400x+x^2+y^2}$

$= \sqrt{200^2-400x+100^2}$

So $F^2 = 200^2+100^2-400x$ and $2F\frac{dF}{dt} = -400\frac{dx}{dt}$.

So $\frac{dF}{dt} = \frac{-200}{F}\,\frac{dx}{dt}$.

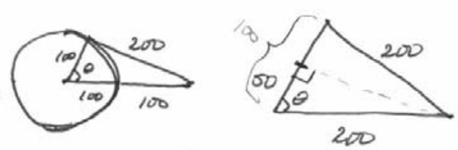
Now $x = 100\cos\theta$ and $7 = \frac{dR}{dt} = 100\,\frac{d\theta}{dt}$.

So $\frac{dx}{dt} = -100\sin\theta\,\frac{d\theta}{dt} = -100\sin\theta\,\frac{7}{100} = -7\sin\theta$

So $\frac{dF}{dt} = \left(\frac{-200}{F}\right)(-7\sin\theta) = \frac{1400\sin\theta}{F}$

So $\left.\frac{dF}{dt}\right|_{F=200} = \left.\frac{1400\sin\theta}{F}\right|_{F=200} = \left.\frac{1400}{200}\sin\theta\right|_{F=200} = \left.7\sin\theta\right|_{F=200}$.

When $F=200$




so $\sin\theta = \frac{\sqrt{200^2-50^2}}{200}$.

So $\left.\frac{dF}{dt}\right|_{F=200} = \frac{7\sqrt{200^2-50^2}}{200} = \frac{7\sqrt{40000-2500}}{200} = \frac{7\sqrt{37500}}{200} = \frac{7}{2}\sqrt{375}$ m/s