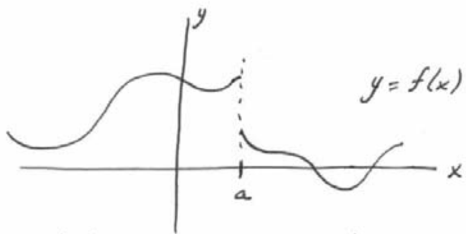
MATH 221 lecture 15, October 11, 2000 ①

A function $f(x)$ is <u>continuous</u> at $x=a$ if it doesn't jump at $x=a$,

i.e. if $\lim_{x\to a} f(x) = f(a)$

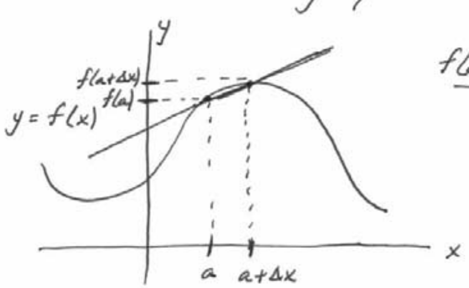


Not continuous at $x=a$.

Think about

$$\left.\frac{df}{dx}\right|_{x=a} = \lim_{\Delta x\to 0} \frac{f(a+\Delta x)-f(a)}{\Delta x}$$

in terms of the graph



$$\frac{f(a+\Delta x)-f(a)}{\Delta x} = \frac{\text{change in } f}{\text{change in } x} = \frac{\text{rise}}{\text{run}}$$

= slope of line connecting $(a, f(a))$ and $(a+\Delta x, f(a+\Delta x))$

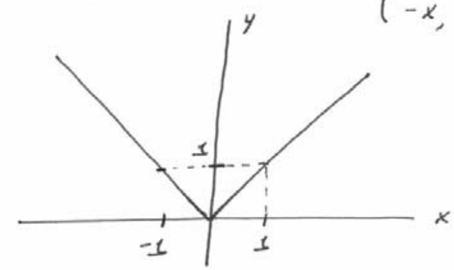
$$\lim_{\Delta x\to 0} \frac{f(a+\Delta x)-f(a)}{\Delta x} = \text{slope of } f \text{ at the point } x=a.$$

②

A function $f(x)$ is <u>differentiable</u> at $x=a$ if the derivative $\left.\frac{df}{dx}\right|_{x=a}$ exists,

i.e. if the slope of the graph of $f(x)$ at $x=a$ exists.

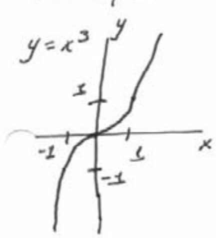
<u>Example</u> Graph $f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$

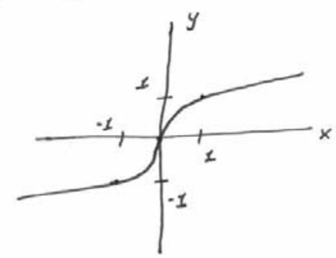


Then $\left.\frac{df}{dx}\right|_{x=a} = \begin{cases} 1, & \text{if } a > 0 \\ -1, & \text{if } a < 0 \\ \text{does not exist}, & \text{if } a = 0 \end{cases}$

So f is not differentiable at $x=0$.

<u>Example</u> Graph $y = x^{\frac{1}{3}}$.





<u>Notes:</u>

(a) $y = x^{\frac{1}{3}}$ is the same as $y^3 = x$.

$\frac{dy}{dx} = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$

So $\left.\frac{dy}{dx}\right|_{x=0} = \infty$. So $f(x)$ is <u>not</u> differentiable at $x=0$.

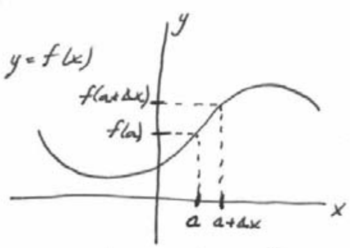
A function $f(x)$ is <u>increasing</u> at $x=a$ if

it is going up at $x=a$,

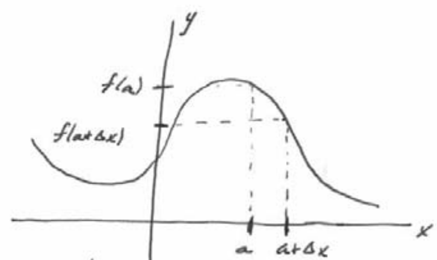
i.e. if $f(a+\Delta x) > f(a)$ for all small $\Delta x > 0$,

i.e. if slope is positive,

i.e. if $\left.\frac{df}{dx}\right|_{x=a} > 0$.



increasing at $x=a$



decreasing at $x=a$

A function $f(x)$ is <u>decreasing</u> at $x=a$ if

it is going down at $x=a$,

i.e. if $f(a+\Delta x) < f(a)$ for all small $\Delta x > 0$,

i.e. if the slope of $f(x)$ at $x=a$ is negative,

i.e. if $\left.\frac{df}{dx}\right|_{x=a} < 0$.

f is <u>concave up</u> at $x=a$ if it is

right side up bowl shaped at $x=a$,

i.e. if the slope of f is getting larger at $x=a$,

i.e. if $\frac{df}{dx}$ is increasing at $x=a$,

i.e. if $\left.\frac{d^2f}{dx^2}\right|_{x=a} > 0$.

f is <u>concave down</u> at $x=a$ if it is

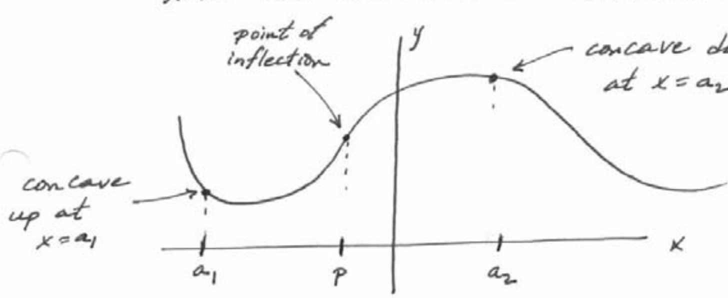
upside down bowl shaped at $x=a$,

i.e. if the slope of f is getting smaller,

i.e. if $\frac{df}{dx}$ is decreasing at $x=a$,

i.e. if $\left.\frac{d^2f}{dx^2}\right|_{x=a} < 0$.

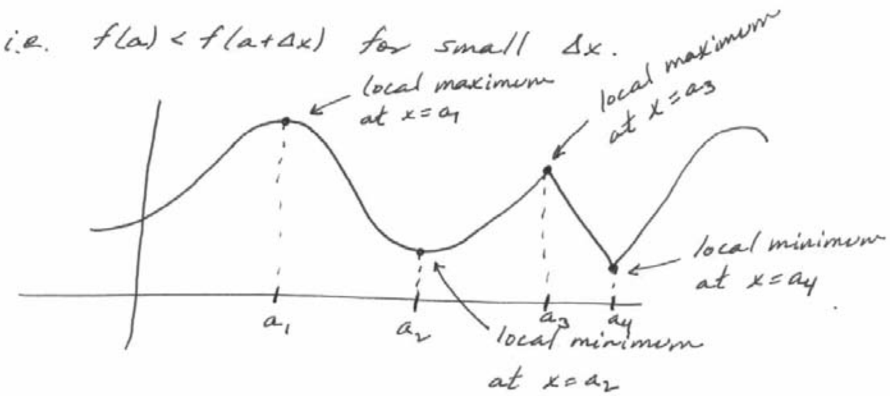
A <u>point of inflection</u> is a point where f changes from concave up to concave down, or from concave down to concave up.



A *local maximum* is a point $x=a$ where $f(a)$ is bigger than the $f(x)$ around it.

A *local minimum* is a point $x=a$ where $f(a)$ is smaller than the $f(x)$ around it.

i.e. $f(a) < f(a+\Delta x)$ for small Δx.



A *critical point* is a point where a maximum or minimum might occur.

Note:

(1) If $f(x)$ is continuous and differentiable and $x=a$ is a maximum then

$$\left.\frac{df}{dx}\right|_{x=a} = 0 \quad \text{and} \quad \left.\frac{d^2f}{dx^2}\right|_{x=a} < 0$$

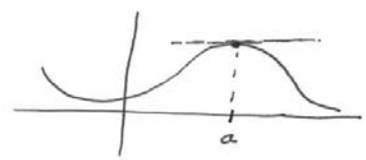
(2) If $f(x)$ is continuous at $x=a$, $f(x)$ is differentiable at $x=a$,

$$\left.\frac{df}{dx}\right|_{x=a} = 0 \quad \text{and} \quad \left.\frac{d^2f}{dx^2}\right|_{x=a} > 0 \quad \underline{\text{then}}$$

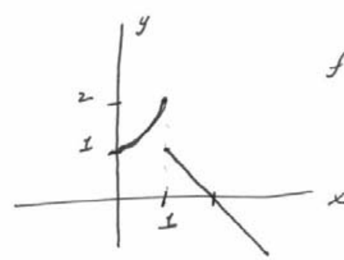
$x=a$ is a minimum.

Where can a maximum or minimum occur?

(a) A point $x=a$ where $f(x)$ is differentiable and $\left.\frac{df}{dx}\right|_{x=a} = 0$.



(b) A point $x=a$ where $f(x)$ is not continuous.



$$f(x) = \begin{cases} x^2+1, & \text{if } 0 \le x \le 1, \\ 2-x, & \text{if } x > 1. \end{cases}$$

$x=1$ is a maximum.