# MATH 221 Lecture 13, October 6, 2000

## Existence of limits

Example What is $\lim_{x \to 0} \frac{1}{x}$?

If $x = .1$ then $\frac{1}{x} = 10$

If $x = .01$ then $\frac{1}{x} = 100$

If $x = .001$ then $\frac{1}{x} = 1000$

If $x = .0001$ Then $\frac{1}{x} = 10000$

So, it looks like $\lim_{x \to 0} \frac{1}{x} = \infty$

If $x = -.1$ then $\frac{1}{x} = -10$

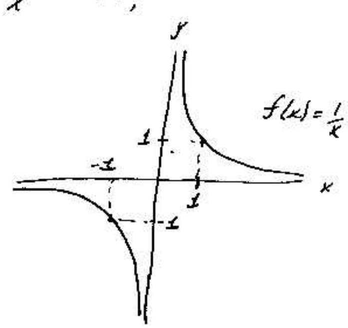
If $x = -.01$ then $\frac{1}{x} = -100$

If $x = -.001$ then $\frac{1}{x} = -1000$

So, it looks like $\lim_{x \to 0} \frac{1}{x} = -\infty$.

Since $\lim_{x \to 0^+} \frac{1}{x} = \infty$ and $\lim_{x \to 0^-} \frac{1}{x} = -\infty$,

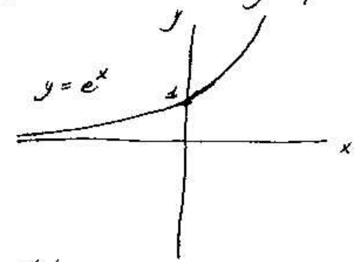
$\lim_{x \to 0} \frac{1}{x} =$ UNDEFINED

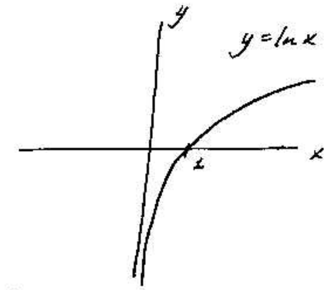





Example $\lim_{x \to -1} \ln x = ???$

Look at the graph of $\ln x$.




Notes:

$e^0 = 1$, $e^1 = 2.718\ldots$

$e^2 \approx 88$, $e^3 \approx 25$

$e^{-1} \approx \frac{1}{3}$, $e^{-2} \approx \frac{1}{9}$

Notes:

$y = \ln x$ means $e^y = x$.

So this graph is the same as the left one but with $x$ and $y$ switched.

So, from the graph, $\ln x$ doesn't even make sense for $x$ close to $-1$. So

$\lim_{x \to -1} \ln x$ is certainly undefined.

Note: If we allow $x$ to get closer and closer to $-1$ and be a complex number then

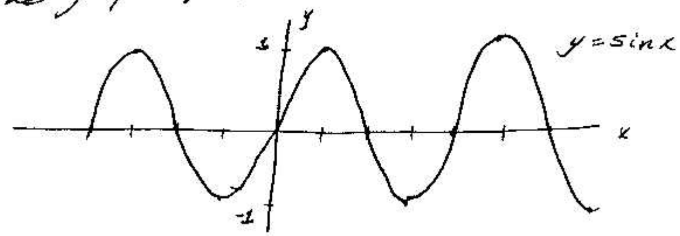
$\ln -1 = i\pi$ and $i3\pi$ and $i5\pi \ldots$

since $e^{i\pi} = \cos\pi + i\sin\pi = -1 + i \cdot 0 = -1$ and $\ln -1 = i\pi$.

Still $\lim_{x \to -1} \ln x$ is undefined since it can't be $i\pi$ and $3i\pi$ and $5i\pi \ldots$ all at once.

Example $\lim\limits_{x \to \infty} \sin x$

The graph of $\sin x$ is




So, as $x$ gets larger and larger, $\sin x$ keeps going back and forth between $-1$ and $+1$.
So $\sin x$ doesn't get closer and closer to anything as $x$ gets larger and larger.

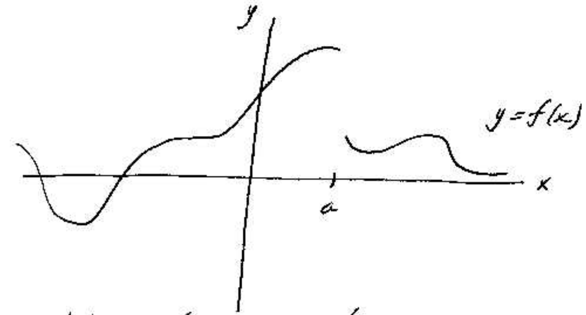
So $\lim\limits_{x \to \infty} \sin x$ is undefined.

Continuous functions

A function is continuous if $f(x)$ doesn't jump when $x$ changes. The function $f(x)$ is not continuous exactly at the places where it jumps.
A function $f(x)$ is continuous at $x = a$ if it doesn't jump at $x = a$,

i.e. if $\lim\limits_{x \to a} f(x) = f(a)$

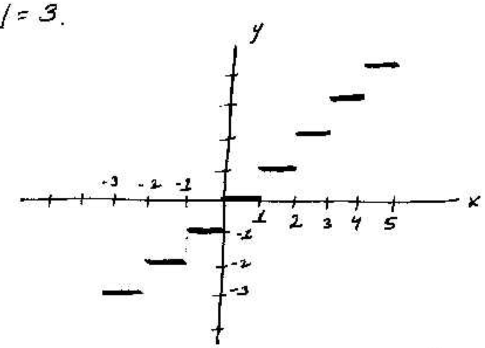





Not continuous at $x = a$.

Example $f(x) = \lfloor x \rfloor$ Round down function

$\lfloor 3.2 \rfloor = 3$.



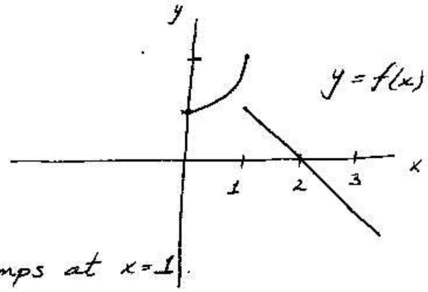

$f(x)$ is continuous if $x \neq 0, \pm 1, \pm 2, \pm 3, \ldots$

Note: $\lim\limits_{x \to 1^-} \lfloor x \rfloor = 0$ and $\lim\limits_{x \to 1^+} \lfloor x \rfloor = 1$.

$f(x) = \lceil x \rceil$ is the round up function

$\lceil 3.2 \rceil = 4$.

Example $f(x) = \begin{cases} 1+x^2, & 0 \le x \le 1, \\ 2-x, & x > 1. \end{cases}$




$f(x)$ jumps at $x = 1$.

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 1 + x^2 = 2.$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2 - x = 1.$$

So $\lim_{x \to 1} f(x)$ is UNDEFINED.

Example: $f(x) = \begin{cases} \dfrac{\sin 3x}{x}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$

$\sin 3x$ is continuous everywhere and $x$ is continuous everywhere,

So $\dfrac{\sin 3x}{x}$ is continuous everywhere

EXCEPT, it makes no sense when $x = 0$.



Now what is happening when $x = 0$?

$$\lim_{x \to 0} \frac{\sin 3x}{x} = \lim_{x \to 0} \frac{\sin 3x}{3x} \cdot 3 = 1 \cdot 3 = 3$$

BUT $f(0) = 1$,

So $\lim_{x \to 0} f(x) \neq f(0)$ in this case.

So $f(x)$ is not continuous when $x = 0$.