MATH 221, Lecture 12, October 4, 2000. ①

Example Evaluate $\lim_{x\to\infty} \frac{x^2-7x+11}{3x^2+10}$

$$\lim_{x\to\infty} \frac{x^2-7x+11}{3x^2+10} = \lim_{x\to\infty} \frac{1-\frac{7}{x}+\frac{11}{x^2}}{3+\frac{10}{x^2}} = \frac{1-0+0}{3+0} = \frac{1}{3}.$$

Example Evaluate $\lim_{x\to 0} \frac{\sin 3x}{\sin 5x}$

$$\lim_{x\to 0} \frac{\sin 3x}{\sin 5x} = \lim_{x\to 0} \frac{\sin 3x}{3x} \cdot 3x \, \frac{5x}{\sin 5x} \, \frac{1}{5x}.$$

$$= \lim_{x\to 0} \frac{\sin 3x}{3x} \, \frac{1}{\frac{\sin 5x}{5x}} \, \frac{3x}{5x} = \lim_{x\to 0} \frac{\sin 3x}{3x} \, \frac{1}{\frac{\sin 5x}{5x}} \cdot \frac{3}{5}$$

$$= 1 \cdot \frac{1}{1} \cdot \frac{3}{5} = \frac{3}{5}.$$

Example Evaluate $\lim_{x\to 1} \frac{1-x}{(\cos^{-1}x)^2}$.

Let $y=\cos^{-1}x$. Then $y\to 0$ as $x\to 1$ and $x=\cos y$.

So

$$\lim_{x\to 1} \frac{1-x}{(\cos^{-1}x)^2} = \lim_{y\to 0} \frac{1-\cos y}{y^2} = \lim_{y\to 0} \frac{(1-\cos y)}{y^2}\frac{(1+\cos y)}{(1+\cos y)}$$

$$= \lim_{y\to 0} \frac{(1-\cos^2 y)}{y^2} \frac{1}{1+\cos y} = \lim_{y\to 0} \frac{\sin y}{y} \frac{\sin y}{y} \frac{1}{1+\cos y} = 1 \cdot 1 \cdot \frac{1}{2} = \frac{1}{2}.$$

②

xample $\lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$ when $f(x)=\sin 2x$.

$$\lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x\to 0} \frac{\sin(2(x+\Delta x)) - \sin 2x}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{\sin(2x+2\Delta x) - \sin 2x}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{\sin 2x \cos 2\Delta x + \cos 2x \sin 2\Delta x - \sin 2x}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \sin 2x \frac{(\cos 2\Delta x - 1)}{\Delta x} + \cos 2x \frac{\sin 2\Delta x}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \sin 2x \frac{\cos 2\Delta x - 1}{2\Delta x} \cdot 2 + \cos 2x \frac{\sin 2\Delta x}{2\Delta x} \cdot 2$$

$$= \sin 2x \cdot 0 \cdot 2 + \cos 2x \cdot 1 \cdot 2 = 2\cos 2x.$$

③

<u>Example</u> $\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$ when $f(x)=\cos x^2$.

$$\lim_{\Delta x \to 0} \frac{\cos(x+\Delta x)^2-\cos x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cos\left(x^2+2x\Delta x+(\Delta x)^2\right)-\cos x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cos x^2 \cos\left(2x\Delta x+(\Delta x)^2\right) - \sin x^2 \sin\left(2x\Delta x+(\Delta x)^2\right) - \cos x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \cos x^2 \frac{\left(\cos\left(2x\Delta x+(\Delta x)^2\right)-1\right)}{\Delta x} - \sin x^2 \frac{\sin\left(2x\Delta x+(\Delta x)^2\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \cos x^2 \frac{\left(\cos\left(2x\Delta x+(\Delta x)^2\right)-1\right)}{2x\Delta x+(\Delta x)^2} \frac{\left(2x\Delta x+(\Delta x)^2\right)}{\Delta x}$$

$$- \sin x^2 \frac{\sin\left(2x\Delta x+(\Delta x)^2\right)}{2x\Delta x+(\Delta x)^2} \frac{\left(2x\Delta x+(\Delta x)^2\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \cos x^2 \frac{\left(\cos(STUFF)-1\right)}{STUFF} \cdot (2x+\Delta x) - \sin x^2 \frac{\sin(STUFF)}{STUFF} \cancel{STUFF} \cdot (2x+\Delta x)$$

$$= \cos x^2 \cdot 0 \cdot 2x - \sin x^2 \cdot 1 \cdot 2x = -2x \sin x^2.$$

④

<u>Example</u> $\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$ when $f(x)=x^x$.

$$\lim_{\Delta x \to 0} \frac{(x+\Delta x)^{x+\Delta x}-x^x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\left(e^{\ln(x+\Delta x)}\right)^{x+\Delta x}-\left(e^{\ln x}\right)^x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{e^{(x+\Delta x)\ln(x+\Delta x)} - e^{x\ln x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} e^{x\ln x} \frac{\left(e^{(x+\Delta x)\ln(x+\Delta x)-x\ln x}-1\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} e^{x\ln x} \frac{\left(e^{(x+\Delta x)\ln(x+\Delta x)-x\ln x}-1\right)}{\left((x+\Delta x)\ln(x+\Delta x)-x\ln x\right)} \frac{\left((x+\Delta x)\ln(x+\Delta x)-x\ln x\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} e^{x\ln x} \left(\frac{e^{STUFF}-1}{STUFF}\right)\left(\frac{x\ln(x+\Delta x)-x\ln x}{\Delta x} + \ln(x+\Delta x)\right)$$

$$= \lim_{\Delta x \to 0} e^{x\ln x} \left(\frac{e^{STUFF}-1}{STUFF}\right)\left(\frac{x\ln\left(x\left(1+\frac{\Delta x}{x}\right)\right)-x\ln x}{\Delta x} + \ln(x+\Delta x)\right)$$

$$= \lim_{\Delta x \to 0} e^{x\ln x} \left(\frac{e^{STUFF}-1}{STUFF}\right)\left(\frac{x\left(\ln x+\ln\left(1+\frac{\Delta x}{x}\right)\right)-x\ln x}{\Delta x} + \ln(x+\Delta x)\right)$$

$$= \lim_{\Delta x \to 0} e^{x\ln x} \left(\frac{e^{STUFF}-1}{STUFF}\right)\left(\frac{x\ln\left(1+\frac{\Delta x}{x}\right)}{\Delta x} + \ln(x+\Delta x)\right)$$

$$= \lim_{\Delta x \to 0} e^{x\ln x} \left(\frac{e^{STUFF}-1}{STUFF}\right)\left(\frac{\ln\left(1+\frac{\Delta x}{x}\right)}{\frac{\Delta x}{x}} + \ln(x+\Delta x)\right) = e^{x\ln x} \cdot 1\,(1+\ln x)$$

$$= x^x + x^x \ln x.$$