MATH 221 Lecture 10, September 29, 2000

Finding derivatives with limits

If f is a function then

$$f(x) = f(a) + \left(\frac{df}{dx}\Big|_{x=a}\right)(x-a) + \frac{1}{2!}\left(\frac{d^2f}{dx^2}\Big|_{x=a}\right)(x-a)^2 +$$

$$+ \frac{1}{3!}\left(\frac{d^3f}{dx^3}\Big|_{x=a}\right)(x-a)^3 + \quad \cdots$$

Substitute $x = a + \Delta x$

Δx stands for a "small change in x".

$$f(a+\Delta x) = f(a) + \left(\frac{df}{dx}\Big|_{x=a}\right)(\Delta x) + \frac{1}{2!}\left(\frac{d^2f}{dx^2}\Big|_{x=a}\right)(\Delta x)^2$$

$$+ \frac{1}{3!}\left(\frac{d^3f}{dx^3}\Big|_{x=a}\right)(\Delta x)^3 + \cdots$$

So

$$f(a+\Delta x) - f(a) = \left(\frac{df}{dx}\Big|_{x=a}\right)\Delta x + \frac{1}{2!}\left(\frac{d^2f}{dx^2}\Big|_{x=a}\right)(\Delta x)^2 + \cdots$$

So $$\frac{f(a+\Delta x) - f(a)}{\Delta x} = \frac{df}{dx}\Big|_{x=a} + \frac{1}{2!}\left(\frac{d^2f}{dx^2}\Big|_{x=a}\right)\Delta x +$$

$$+ \frac{1}{3!}\left(\frac{d^3f}{dx^3}\Big|_{x=a}\right)(\Delta x)^2 + \cdots$$

So

$$\lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x} = \frac{df}{dx}\Big|_{x=a}.$$

Example Suppose $f(x) = x^3$. What is $f(3.02)$?

$$f(3.02) = f(3 + .02)$$

$$= f(3) + \left(\frac{df}{dx}\Big|_{x=3}\right)(.02) + \frac{1}{2}\left(\frac{d^2f}{dx^2}\right)(.02)^2 + \cdots$$

since

$$f(a+\Delta x) = f(a) + \left(\frac{df}{dx}\Big|_{x=a}\right)\Delta x + \frac{1}{2}\left(\frac{d^2f}{dx^2}\Big|_{x=a}\right)(\Delta x)^2 + \cdots$$

Now

$$f(3) = 27, \quad \frac{df}{dx}\Big|_{x=3} = 3x^2\Big|_{x=3} = 27,$$

$$\frac{d^2f}{dx^2}\Big|_{x=3} = 6x\Big|_{x=3} = 18, \quad \frac{d^3f}{dx^3}\Big|_{x=3} = 6\Big|_{x=3} = 6$$

$$\frac{d^4f}{dx^4}\Big|_{x=3} = 0\Big|_{x=3} = 0, \quad \frac{d^5f}{dx^5}\Big|_{x=3} = 0\Big|_{x=3} = 0, \quad \ldots$$

So $$f(3.02) = f(3 + .02)$$

$$= 27 + 27(.02) + \tfrac{1}{2}18(.02)^2 + \frac{6}{3!}(.02)^3 + 0 + 0 + \cdots$$

$$= 27 + .54 + 9(.0004) + .000008$$

$$= 27.543608.$$

Example What is the expansion of $f(a+\Delta x)$ ③

when $f(x)=e^{3x}$ and $a=0$?

$$f(a+\Delta x) = e^{3(a+\Delta x)} = e^{3\Delta x}$$

$$= 1 + 3\Delta x + \frac{(3\Delta x)^2}{2!} + \frac{(3\Delta x)^3}{3!} + \frac{(3\Delta x)^4}{4!} + \cdots$$

$$= 1 + 3\Delta x + \frac{9}{2!}(\Delta x)^2 + \frac{27}{3!}(\Delta x)^3 + \frac{3^4(\Delta x)^4}{4!} + \cdots$$

Second way:

$$f(a+\Delta x) = f(0+\Delta x)$$

$$= f(0) + \left.\frac{df}{dx}\right|_{x=0} \Delta x + \frac{1}{2}\left(\left.\frac{d^2 f}{dx^2}\right|_{x=0}\right)(\Delta x)^2 + \cdots$$

$$f(0) = e^{3\cdot 0} = 1, \quad \left.\frac{df}{dx}\right|_{x=0} = \left.\frac{de^{3x}}{dx}\right|_{x=0} = \left.3e^{3x}\right|_{x=0} = 3,$$

$$\left.\frac{d^2 f}{dx^2}\right|_{x=0} = \left.\frac{d\, 3e^{3x}}{dx}\right|_{x=0} = \left.3^2 e^{3x}\right|_{x=0} = 3^2,$$

$$\left.\frac{d^3 f}{dx^3}\right|_{x=0} = \left.\frac{d\, 3^2 e^{3x}}{dx}\right|_{x=0} = \left.3^3 e^{3x}\right|_{x=0} = 3^3, \quad \ldots$$

So

$$f(0+\Delta x) = e^{3\Delta x} = 1 + 3\Delta x + \frac{1}{2} 3^2 (\Delta x)^2 + \frac{1}{3!} 3^3 (\Delta x)^3 + \cdots$$

Example If $f(x) = \ln(1+x)$ expand $f(a+\Delta x)$ in terms of Δx when $a=0$. ④

$$f(a+\Delta x) = f(0+\Delta x) = f(\Delta x) = \ln(1+\Delta x)$$

$$= f(0) + \left(\left.\frac{df}{dx}\right|_{x=0}\right)\Delta x + \frac{1}{2}\left(\left.\frac{d^2 f}{dx^2}\right|_{x=0}\right)(\Delta x)^2 + \frac{1}{3!}\left(\left.\frac{d^3 f}{dx^3}\right|_{x=0}\right)(\Delta x)^3 + \cdots$$

$$= \ln(1+0) + \left.\frac{d\ln(1+x)}{dx}\right|_{x=0} \Delta x + \frac{1}{2!}\left(\left.\frac{d^2 \ln(1+x)}{dx^2}\right|_{x=0}\right)(\Delta x)^2 + \cdots$$

$$\ln(1+0) = \ln 1 = 0$$

$$\left.\frac{d\ln(1+x)}{dx}\right|_{x=0} = \left.\frac{1}{1+x}\right|_{x=0} = \frac{1}{1} = 1.$$

$$\left.\frac{d^2 \ln(1+x)}{dx^2}\right|_{x=0} = \left.\frac{d\frac{1}{1+x}}{dx}\right|_{x=0} = \left.\frac{-1}{(1+x)^2}\right|_{x=0} = -1.$$

$$\left.\frac{d^3 \ln(1+x)}{dx^3}\right|_{x=0} = \left.\frac{d\frac{-1}{(1+x)^2}}{dx}\right|_{x=0} = \left.\frac{(-1)(-2)}{(1+x)^3}\right|_{x=0} = 2\cdot 1$$

$$\left.\frac{d^4 \ln(1+x)}{dx^4}\right|_{x=0} = \left.\frac{d\frac{2!}{(1+x)^3}}{dx}\right|_{x=0} = \left.\frac{-3\cdot 2\cdot 1}{(1+x)^4}\right|_{x=0} = -3!$$

So

$$\ln(1+\Delta x) = 0 + (+1)\Delta x + \frac{1}{2!}(-1)(\Delta x)^2 + \frac{1}{3!}2!(\Delta x)^3 + \frac{1}{4!}(-3!)(\Delta x)^4 + \cdots$$

$$= 0 + \Delta x - \frac{(\Delta x)^2}{2} + \frac{(\Delta x)^3}{3} - \frac{(\Delta x)^4}{4} + \cdots$$

$$= \Delta x - \frac{(\Delta x)^2}{2} + \frac{(\Delta x)^3}{3} - \frac{(\Delta x)^4}{4} + \cdots$$

The *linear* approximation to $\ln(1+x)$ at $x=0$ is

$$\ln(1+\Delta x) \cong +\Delta x$$

The *quadratic* approximation to $\ln(1+x)$ at $x=0$ is

$$\ln(1+\Delta x) \cong \Delta x - \frac{(\Delta x)^2}{2!}$$

Example Approximate the value of $(255)^{\frac{1}{4}}$.

Let $f(x) = x^{\frac{1}{4}}$. Then

$$(255)^{\frac{1}{4}} = (256-1)^{\frac{1}{4}} = f(a+\Delta x) \quad \text{with } a = 256 \text{ and } \Delta x = -1.$$

$$f(a+\Delta x) \cong f(a) + \left(\frac{df}{dx}\bigg|_{x=a}\right)\Delta x = f(256) + \left(\frac{df}{dx}\bigg|_{x=256}\right)(-1)$$

$$= (256)^{\frac{1}{4}} + \left(\frac{1}{4}x^{-3/4}\bigg|_{x=256}\right)(-1)$$

$$= 4 + \frac{1}{4}(256)^{-3/4}(-1) = 4 + \frac{1}{4}4^{-3}(-1)$$

$$= 4 + \frac{1}{4}\frac{1}{4^3}(-1) = 4 - \frac{1}{256} = 3\frac{255}{256} = 3.99609375$$

This is an approximation to $255^{\frac{1}{4}}$ using a linear approximation.

$$f(a+\Delta x) \cong f(a) + \left(\frac{df}{dx}\bigg|_{x=a}\right)\Delta x + \frac{1}{2}\left(\frac{d^2f}{dx^2}\bigg|_{x=a}\right)(\Delta x)^2$$

$$= a^{\frac{1}{4}} + \left(\frac{1}{4}x^{-\frac{3}{4}}\bigg|_{x=a}\right)\Delta x + \frac{1}{2}\left(\left(\frac{1}{4}\right)\left(\frac{-3}{4}\right)x^{-7/4}\bigg|_{x=a}\right)(\Delta x)^2$$

$$= a^{\frac{1}{4}} + \frac{1}{4}(a^{\frac{1}{4}})^{-3}\Delta x + \frac{-3}{2}\frac{1}{4^2}(a^{\frac{1}{4}})^{-7}(\Delta x)^2$$

$$= 4 + \frac{1}{4}4^{-3}(-1) - \frac{3}{2\cdot 4^2}4^{-7}(-1)^2$$

$$= 4 - \frac{1}{4^4} + \frac{3}{2\cdot 4^9} = 4 - \frac{1}{16} + \frac{3}{2\cdot 262144}$$

$$= 3.99609947204589843750$$

This is the quadratic approximation to $255^{\frac{1}{4}}$.
The correct answer is

$$255^{\frac{1}{4}} = 3.9960880148804670689\ldots$$

according to my computer. The computer is clearly wrong (it is off by at least .000005).

Example Expand $f(a+\Delta x)$ in terms of Δx when $a = 4$ and $f(x) = \sqrt{x}$.

$$f(a+\Delta x) = f(4+\Delta x) = \sqrt{4+\Delta x}$$

$$= f(4) + \left(\frac{df}{dx}\bigg|_{x=4}\right)\Delta x + \frac{1}{2!}\left(\frac{d^2f}{dx^2}\bigg|_{x=4}\right)(\Delta x)^2 + \ldots$$

$f(4) = \sqrt{4} = 2.$

$$\frac{df}{dx}\bigg|_{x=4} = \frac{d\,x^{\frac{1}{2}}}{dx}\bigg|_{x=4} = \frac{1}{2}x^{-\frac{1}{2}}\bigg|_{x=4} = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$$

$$\left.\frac{d^2f}{dx^2}\right|_{x=4} = \left.\frac{d\,\frac{1}{2}x^{-\frac{1}{2}}}{dx}\right|_{x=4} = \left.\frac{1}{2}\left(\frac{-1}{2}\right)x^{-3/2}\right|_{x=4} = \frac{-1}{2\cdot 2}\,\frac{1}{2^3} = \frac{-1}{2^5}$$

$$\left.\frac{d^3f}{dx^3}\right|_{x=4} = \left.\frac{d\,\frac{-1}{2^2}x^{-3/2}}{dx}\right|_{x=4} = \left.\frac{-1}{2^2}\,\frac{-3}{2}\,x^{-5/2}\right|_{x=4} = \frac{3}{2^3}\,\frac{1}{2^5} = \frac{3}{2^8}$$

$$\left.\frac{d^4f}{dx^4}\right|_{x=4} = \left.\frac{d\,\frac{3}{2^3}x^{-5/2}}{dx}\right|_{x=4} = \left.\frac{3}{2^3}\left(\frac{-5}{2}\right)x^{-7/2}\right|_{x=4} = \frac{-3\cdot 5}{2^4}\,\frac{1}{2^7} = \frac{-3\cdot 5}{2^{11}}$$

$\cdots$

So

$$\sqrt{4+\Delta x} = f(4+\Delta x)$$

$$= 2 + \frac{1}{2^2}\Delta x - \frac{1}{2!}\,\frac{1}{2^5}(\Delta x)^2 + \frac{1}{3!}\,\frac{3}{2^8}(\Delta x)^3 - \frac{1}{4!}\,\frac{3\cdot 5}{2^{11}}(\Delta x)^4 + \cdots$$

The linear approximation to $\sqrt{x}$ at $x=4$ is

$$\sqrt{4+\Delta x} \approx 2 + \frac{1}{4}\Delta x$$

The quadratic approximation to $\sqrt{x}$ at $x=4$ is

$$\sqrt{4+\Delta x} \approx 2 + \frac{1}{4}\Delta x - \frac{1}{64}(\Delta x)^2.$$

<u>Example</u> Approximate $\sqrt{4.03}$.

<u>Linear:</u>

$\sqrt{4.03} \approx 2+\frac{1}{4}(.03) = 2.0075,$

<u>Quadratic:</u>

$\sqrt{4.03} \approx 2+\frac{1}{4}(.03) - \frac{1}{64}(.03)^2$

$= 2.00748593 75$