MATH 221: Calculus and Analytic Geometry
Prof. Ram, Fall 2006

HOMEWORK 8
DUE October 30, 2006

Problem A. Tangents and normals.

(1) Find the equations of the tangent and normal to the curve $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at the point where $x = 1$.

(2) Find the equations of the tangent and normal to the curve $y = \cot^2 x - 2\cot x + 2$ at $x = \pi/4$.

(3) Find the equation of the tangent to the curve given by the equations $x = \theta + \sin\theta$ and $y = 1 + \cos\theta$ at $\theta = \pi/4$.

(4) Find the equation of the tangent to the curve given by the equations $x = a\cos\theta$ and $y = b\sin\theta$ at $\theta = \pi/4$.

(5) For a general t find the equation of the tangent and normal to the curve given by the equations $x = a\cos t$ and $y = b\sin t$.

(6) For a general t find the equation of the tangent and normal to the curve $x = a\sec t$, $y = b\tan t$.

(7) For a general t find the equation of the tangent and normal to the curve given by the equations $x = a(t + \sin t)$ and $y = b(1 - \cos t)$.

(8) Find the equations of the tangent and normal to the curve $16x^2 + 9y^2 = 144$ at (x_1, y_1) where $x_1 = 2$ and $y_1 > 0$.

(9) Find the equation of the tangent to the curve $y = \sec^4 x - \tan^4 x$ at $x = \pi/3$.

(10) Find the equation of the normal to the curve $y = (\sin 2x + \cot x + 2)^2$ at $x = \pi/2$.

(11) Find the equation of the normal to the curve $y = \dfrac{1 + \sin x}{\cos x}$ at $x = \pi/4$.

(12) Show that the tangents to the curve $y = 2x^3 - 3$ at the points where $x = 2$ and $x = -2$ are parallel.

(13) Show that the tangents to the curve $y = x^2 - 5x + 6$ at the points $(2, 0)$ and $(3, 0)$ are at right angles.

(14) Find the points on the curve $2a^2y = x^3 - 3ax^2$ where the tangent is parallel to the x-axis.

(15) For the curve $y(x-2)(x-3) = x - 7$ show that the tangent is parallel to the x-axis at the points for which $x = 7 \pm 2\sqrt{5}$.

(16) Find the points on the curve $y = 4x^3 - 2x^5$ for which the tangent passes through the origin.

(17) Find the points on the circle $x^2 + y^2 = 13$ where the tangent is parallel to the line $2x + 3y = 7$.

(18) Find the point on the curve $y = 3x^2 + 4$ at which the tangent is perpendicular to the line whose slope is $-1/6$.

(19) Find the equations of the normal to the curve $2x^2 - y^2 = 14$ parallel to the line $x + 3y = 4$.

(20) Find the equation of the tangent to the curve $x^2 + 2y = 8$ which is perpendicular to the line $x - 2y + 1 = 0$.

(21) (Bonus problem) If the straight line $x\cos\alpha + y\sin\alpha = p$ touches the curve $\left(\frac{x}{a}\right)^{n/(n-1)} + \left(\frac{y}{b}\right)^{n/(n-1)} = 1$ show that $(a\cos\alpha)^n + (b\sin\alpha)^n = p^n$.

Problem B. Optimization.

(1) Find the local maxima and minima of $f(x) = (5x-1)^2 + 4$ without using derivatives.

(2) Find the local maxima and minima of $f(x) = -(x-3)^2 + 9$ without using derivatives.

(3) Find the local maxima and minima of $f(x) = -|x+4| + 6$ without using derivatives.

(4) Find the local maxima and minima of $f(x) = \sin 2x + 5$ without using derivatives.

(5) Find the local maxima and minima of $f(x) = |\sin 4x + 3|$ without using derivatives.

(6) Find the local maxima and minima of $f(x) = x^4 - 62x^2 + 120x + 9$.

(7) Find the local maxima and minima of $f(x) = (x-1)(x+2)^2$.

(8) Find the local maxima and minima of $f(x) = -(x-1)^3(x+1)^2$.

(9) Find the local maxima and minima of $f(x) = x/2 + 2/x$ for $x > 0$.

(10) Find the local maxima and minima of $f(x) = 2x^3 - 24x + 107$ in the interval $[1, 3]$.

(11) Find the local maxima and minima of $f(x) = \sin x + (1/2)\cos x$ in $0 \leq x \leq \pi/2$.

(12) Show that the maximum value of $\left(\dfrac{1}{x}\right)^x$ is $e^{1/e}$.

(13) Show that $f(x) = x + 1/x$ has a local maximum and a local minimum, but the maximum value is less than the minimum value.

(14) Find the maximum profit that a company can make if the profit function is given by $p(x) = 41 + 24x - 18x^2$.

(15) An enemy jet is flying along the curve $y = x^2 + 2$. A soldier is placed at the point $(3, 2)$. At what point will the jet be at when the soldier and the jet are closest?

(16) Find the local maxima and minima of $f(x) = -x + 2\sin x$ in $[0, 2\pi]$.

(17) Divide 15 into two parts such that the square of one times the cube of the other is maximum.

(18) Suppose the sum of two numbers is fixed. Show that their product is maximum exactly when each one of them is half of the total sum.

(19) Divide a into two parts such that the pth power of one times the qth power of the other is maximum.

(20) Which fraction exceeds its pth power by the maximum amount?

(21) Find the dimensions of the rectangle of area 96 cm^2 which has minimum perimeter. What is this minimum perimeter?

(22) Show that the right circular cone with a given volume and minimum curved surface area has altitude equal to $\sqrt{2}$ times the radius of the base.

(23) Show that the altitude of the right circular cone with maximum volume that can be inscribed in a sphere of radius R is $4R/3$.

(24) Show that the height of a right circular cylinder with maximum volume that can be inscribed in a given right circular cone of height h is $h/3$.

(25) A cylindrical can is to be made to hold 1 liter of oil. Find the dimensions of the can which will minimize the cost of the metal to make the can.

(26) An open box is to be made out of a given quantity of cardboard of area p^2. Find the maximum volume of the box if its base is square.

(27) Show that $f(x) = \sin x(1 + \cos x)$ is maximum when $x = \pi/3$.

(28) An 8 inch piece of wire is to be cut into two pieces. Figure out where to cut the wire in order to make the sum of the squares of the lengths of the two pieces as small as possible.

(29) Find the dimensions of the maximum rectangular area that can be fenced with a fence 300 yards long.

(30) Given the perimeter of a rectangle show that its diagonal is minimum when it is a square. Make up a word problem for which this gives the solution.

(31) Prove that the rectangle of maximum area that can be inscribed in a circle is a square. Make up a word problem for which this gives the solution.

(32) Show that the triangle of the greatest area with given base and vertical angle is isosceles.

(33) Show that a right triangle with a given perimeter has greatest area when it is isosceles.

(34) Show that the angle of the cone with a given slant height and with maximum volume is $\tan^{-1}(\sqrt{2})$.

Problem C. Related rates.

(1) Find the rate of change of the volume of a sphere of radius r with respect to a change in the radius.

(2) Find the rate of change of the volume of a cylinder of radius r and height h with respect to a change in the radius.

(3) Find the rate of change of the curved surface of a cone of radius r and height h with respect to a change in the radius.

(4) The side of a square is increasing at the rate of 0.2 cm/s. Find the rate of increase of the perimeter of the square.

(5) A balloon which always remains spherical is being inflated by pumping in 900 cubic centimeters of gas per second. Find the rate at which the radius of the balloon is increasing when the radius is 15cm.

(6) The surface area of a spherical bubble is increasing at 2 cm^2/s. When the radius of the bubble is 6 cm at what rate is the volume of the bubble increasing?

(7) The bottom of a rectangular swimming pool is 25×40 meters. Water is pumped into the tank at the rate of 500 cubic meters per minute. Find the rate at which the level of the water in the tank is rising.

(8) A runner runs around a circular track of radius 100 m at a constant speed of 7 m/s. The runner's friend is standing 200 m from the center of the track. How fast is the distance between the friends changing when the distance between them is 200 m.

(9) A streetlight is at the top of a 15 foot tall pole. A man 6 ft tall walks away from the pole with a speed of 5 ft/s along a straight path.
  (a) How fast is the tip of his shadow moving when he is 40 feet from the pole?
  (b) How fast is his shadow lengthening at that point?

(10) A lighthouse is on a small island 3 km away from the nearest point P on a straight shoreline and its light turns four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from P?

(11) A boat is pulled into a dock by a rope attached to the bow of the boat a passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/s how fast is the boat approaching the dock when it is 8 m from the dock?

(12) Gravel is being dumped from a conveyor belt at a rate of 30 ft^3/min and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10 ft high?

(13) Water is dripping from a tiny hole in the vertex in the bottom of a conical funnel at a uniform rate of 4 cm^3/s. When the slant height of the water is 3 cm, find the rate of decrease of the slant height of the water, given that the vertical angle of the funnel is $120°$.

(14) Water is leaking out of an inverted conical tank at a rate of $10,000$ cm^3/min at the same time that water is being pumped into the tank at a constant rate. The tank has height 6 m and the diameter at the top is 4 m. If the water height is rising at a rate of 20 cm/min when the height of the water is 2 m, find the rate at which water is being pumped into the tank.

(15) Oil is leaking from a cylindrical drum at a rate of 16 milliliters per second. If the radius of the drum is 7 cm and its height is 60 cm find the rate at which the level of oil is changing when the oil level is 18 cm.

(16) A ladder 10 feet long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a speed of 2 ft/s, how fast is the angle between the top of the ladder and the wall changing when the angle is $\pi/4$ radians?

(17) A ladder 13 meters long is leaning against a wall. The bottom of the ladder is pulled along the ground away from from the wall at the rate of 2 m/s. How fast is its height on the wall decreasing when the foot of the ladder is 5 m away from the wall?

(18) A man is moving away from a 40 meter tower at a speed of 2 m/s. Find the rate at which the angle of elevation of the top of the tower is changing when he is at a distance of 30 meters from the foot of the tower. Assume that the eye level of the man is 1.6 meters from the ground.

(19) Find the angle which increases twice as fast as its sine.

(20) A television camera is positioned 4000 ft from the base of a rocket launching pad. A rocket rises vertically and its speed is 600 ft/s when it has risen 3000 feet.
   (a) How fast is the distance from the television camera to the rocket changing at that moment?
   (b) How fast is the camera's angle of elevation changing at that same moment?