Math 521 Lecture 3: Homework 9
Fall 2004, Professor Ram
Due December 15, 2004

1 Exercises

1. Give an example of a doubly indexed "sequence" $(a_{m,n})$ such that
$$\lim_{m\to\infty}\lim_{n\to\infty} a_{m,n} \neq \lim_{n\to\infty}\lim_{m\to\infty} a_{m,n}.$$
2. Give an example of a sequence of functions $(f_1, f_2, \ldots)$ with a point x such that
$$\lim_{n\to\infty}\lim_{t\to x} f_n(t) \neq \lim_{t\to x}\lim_{n\to\infty} f_n(t),$$
even though all limits in tis expression exist.
3. Give an example of a sequence $(f_1, f_2, \ldots)$ of differentiable functions such that
$$f = \lim_{n\to\infty} f_n \qquad \text{exists and is differentiable,}$$
but
$$f' \neq \lim_{n\to\infty} f_n'.$$
4. Give an example of a sequence of continuous functions $f_n(x)$ such that
$$\sum_{n\in\mathbb{Z}_{>0}} f_n \quad \text{exists} \qquad \text{but is not continuous.}$$
5. Show that a uniformly convergent sequence of functions is pointwise convergent.
6. Give and example to show that a pointwise convergent sequence of functions is not necessarily uniformly convergent.
7. Show that a sequence $(f_1, f_2, \ldots)$ converges in $C(X)$ if and only it $(f_1, f_2, \ldots)$ is uniformly convergent.
8. Show that if a $(f_1, f_2, \ldots)$ is a sequence in $C(X)$ which converges then the limit function is an element of $C(X)$.
9. Show that if $(f_1, f_2, \ldots)$ is a uniformly convergent sequence then
$$\lim_{n\to\infty}\lim_{t\to x} f_n(t) = \lim_{t\to x}\lim_{n\to\infty} f_n(t).$$
First state the theorem precisely.

10. Show that $C(X)$ is a metric space.
11. Given an example of a Cauchy sequence in $C(X)$.
12. Show that $C(X)$ is a complete metric space.
13. Show that $C(X)$ is an algebra.
14. Let $\mathcal{U}(X)$ be the set of functions f on X such that the sup norm of f exists. Give an example of an element of $\mathcal{M}(X)$ that is not an element of $\mathcal{U}(X)$.
15. Let $\mathcal{U}(X)$ be the set of functions f on X such that the sup norm of f exists. Is $\mathcal{U}(X)$ a complete metric space?
16. Prove the Weierstrass theorem.
17. Discuss the relation between the Weierstrass theorem and the Taylor theorem.
18. Prove the Stone theorem.
19. Discuss the relation between the Stone theorem and the Weierstrass theorem.