§1P. Fields

(3.1.3) Proposition. *If $f: K \to F$ is a field homomorphism then f is injective.*

Proof.

To show: $f: K \to F$ is injective.

Assume $f: K \to F$ is a field homomorphism.

To show: If $x_1, x_2 \in K$ and $f(x_1) = f(x_2)$ then $x_1 = x_2$.

Assume $x_1, x_2 \in K$ and $f(x_1) = f(x_2)$.

To show: $x_1 = x_2$.

Proof by contradiction: Assume $x_1 \neq x_2$.

Let 0_K and 0_F be the additive identities in K and F respectively.

Let $1_K$and 1_F be the multiplicative identities in K and F respectively.

Then $f(x_1) - f(x_2) = 0_F$ and $x_1 - x_2 \neq 0_K$.

Let $y = (x_1 - x_2)^{-1}$, which exists by property h) in the definition of a field.

Then, since $f: K \to F$ is a homomorphism and $f(x_1) - f(x_2) = 0_F$,

$$\begin{aligned} 1_F = f(1_K) &= f\big((x_1 - x_2)y\big) \\ &= f(x_1 - x_2)f(y) \\ &= \big(f(x_1) - f(x_2)\big)f(y) \\ &= 0_F \cdot f(y) \\ &= 0_F. \end{aligned}$$

This is a contradiction to property g) in the definition of a field.

So $x_1 = x_2$.

So $f: K \to F$ is injective. □

(3.2.4) Proposition. *Let V be a vector space over a field F and let W be a subgroup of V. Then the cosets of W in V partition V.*

Proof.

To show: a) If $v \in V$ then $v \in v' + W$ for some $v' \in V$.
b) If $(v_1 + W) \cap (v_2 + W) \neq \emptyset$ then $v_1 + W = v_2 + W$.

a) Let $v \in V$.
Then, since $0 \in W$, $v = v + 0 \in v + W$.
So $v \in v + W$.

b) Assume $(v_1 + W) \cap (v_2 + W) \neq \emptyset$.
To show: ba) $v_1 + W \subseteq v_2 + W$.
bb) $v_2 + W \subseteq v_1 + W$.
Let $a \in (v_1 + W) \cap (v_2 + W)$.
Suppose $a = v_1 + w_1$ and $a = v_2 + w_2$ where $w_1, w_2 \in W$.
Then

$$v_1 = v_1 + w_1 - w_1 = a - w_1 = v_2 + w_2 - w_1 \quad \text{and}$$
$$v_2 = v_2 + w_2 - w_2 = a - w_2 = v_1 + w_1 - w_2.$$

ba) Let $v \in v_1 + W$.
Then $v = v_1 + w$ for some $w \in W$.
Then

$$v = v_1 + w = v_2 + w_2 - w_1 + w \in v_2 + W,$$

since $w_2 - w_1 + w \in W$.
So $v_1 + W \subseteq v_2 + W$.

bb) Let $v \in v_2 + W$.
Then $v = v_2 + w$ for some $w \in W$.
Then

$$v = v_2 + w = v_1 + w_1 - w_2 + w \in v_1 + W,$$

since $w_1 - w_2 + w \in W$.
So $v_2 + W \subseteq v_1 + W$.

So $v_1 + W = v_2 + W$.

So the cosets of W in V partition V. □

(3.2.5) Theorem. *Let W be a subgroup of a vector space V over a field F. Then W is a subspace of V if and only if V/W with operations given by*

$$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W, \quad \textit{and}$$
$$c(v + W) = cv + W,$$

is a vector space over F.

Proof.

$\Longrightarrow$: Assume W is a subspace of V.

To show: a) $(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$ is a well defined operation on V/W.
b) The operation given by $c(v + W) = cv + W$ is well defined.
c) $\big((v_1 + W) + (v_2 + W)\big) + (v_3 + W) = (v_1 + W) + \big((v_2 + W) + (v_3 + W)\big)$ for all $v_1 + W, v_2 + W, v_3 + W \in V/W$.
d) $(v_1 + W) + (v_2 + W) = (v_2 + W) + (v_1 + W)$ for all $v_1 + W, v_2 + W \in V/W$.

e) $0 + W = W$ is the zero in V/W.
f) $-v + W$ is the additive inverse of $v + W$.
g) If $c_1, c_2 \in F$ and $v + W \in V/W$, then $c_1\big(c_2(v+W)\big) = (c_1c_2)(v+W)$.
h) If $v + W \in V/W$ then $1(v+W) = v + W$.
i) If $c \in F$ and $v_1 + W, v_2 + W \in V/W$,
then $c\big((v_1+W)+(v_2+W)\big) = c(v_1+W) + c(v_2+W)$.
j) If $c_1, c_2 \in F$ and $v + W \in V/W$,
then $(c_1+c_2)(v+W) = c_1(v+W) + c_2(v+W)$.

a) We want the operation on V/W given by

$$\begin{array}{ccc} V/W \times V/W & \to & V/W \\ (v_1+W, v_2+W) & \mapsto & (v_1+v_2)+W \end{array}$$

to be well defined.
Let $(v_1+W, v_2+W), (v_3+W, v_4+W) \in V/W \times V/W$ such that $(v_1+W, v_2+W) = (v_3+W, v_4+W)$.
Then $v_1 + W = v_3 + W$ and $v_2 + W = v_4 + W$.
To show: $(v_1+v_2)+W = (v_3+v_4)+W$.
So we must show: aa) $(v_1+v_2)+W \subseteq (v_3+v_4)+W$.
ab) $(v_3+v_4)+W \subseteq (v_1+v_2)+W$.

aa) We know $v_1 = v_1 + 0 \in v_3 + W$ since $v_1 + W = v_3 + W$.
So $v_1 = v_3 + w_1$ for some $w_1 \in W$.
Similarly $v_2 = v_4 + w_2$ for some $w_2 \in W$.
Let $t \in (v_1+v_2)+W$.
Then $t = v_1 + v_2 + w$ for some $w \in W$.
So

$$\begin{aligned} t &= v_1 + v_2 + w \\ &= v_3 + w_1 + v_4 + w_2 + w \\ &= v_3 + v_4 + w_1 + w_2 + w, \end{aligned}$$

since addition is commutative.
So $t = (v_3+v_4) + (w_1+w_2+w) \in v_3 + v_4 + W$.
So $(v_1+v_2)+W \subseteq (v_3+v_4)+W$.

ab) Since $v_1 + W = v_3 + W$, we know $v_1 + w_1 = v_3$ for some $w_1 \in W$.
Since $v_2 + W = v_4 + W$, we know $v_2 + w_2 = v_4$ for some $w_2 \in W$.
Let $t \in (v_3+v_4)+W$.
Then $t = v_3 + v_4 + w$ for some $w \in W$.
So

$$\begin{aligned} t &= v_3 + v_4 + w \\ &= v_1 + w_1 + v_2 + w_2 + w \\ &= v_1 + v_2 + w_1 + w_2 + w, \end{aligned}$$

since addition is commutative.
So $t = (v_1+v_2) + (w_1+w_2+w) \in (v_1+v_2)+W$.
So $(v_3+v_4)+W \subseteq (v_1+v_2)+W$.

So $(v_1+v_2)+W = (v_3+v_4)+W$.
So the operation given by $(v_1+W)+(v_3+W) = (v_1+v_3)+W$ is a well defined operation on V/W.

b) We want the operation given by

$$\begin{array}{ccc} F \times V/W & \to & V/W \\ (c, v+W) & \mapsto & cv+W \end{array}$$

to be well defined.
Let $(c_1, v_1 + W), (c_2, v_2 + W) \in (F \times V/W)$ such that $(c_1, v_1 + W) = (c_2, v_2 + W)$.
Then $c_1 = c_2$ and $v_1 + W = v_2 + W$.
To show: $c_1 v_1 + W = c_2 v_2 + W$.

To show: ba) $c_1 v_1 + W \subseteq c_2 v_2 + W$.
bb) $c_2 v_2 + W \subseteq c_1 v_1 + W$.

ba) Since $v_1 + W = v_2 + W$, we know $v_1 = v_2 + w_1$ for some $w_1 \in W$.
Let $t \in c_1 v_1 + W$.
Then $t = c_1 v_1 + w$ for some $w \in W$. So

$$\begin{aligned} t &= c_1 v_1 + w \\ &= c_2(v_2 + w_1) + w \\ &= c_2 v_2 + c_2 w_1 + w, \end{aligned}$$

since $c_1 = c_2$.
Since W is a subspace, $c_2 w_1 \in W$, and $c_2 w_1 + w \in W$.
So $t = c_2 v_2 + c_2 w_1 + w \in c_2 v_2 + W$.
So $c_1 v_1 + W \subseteq c_2 v_2 + W$.

bb) Since $v_1 + W = v_2 + W$, we know $v_2 = v_1 + w_2$ for some $w_2 \in W$.
Let $t \in c_2 v_2 + W$.
Then $t = c_2 v_2 + w$ for some $w \in W$. So

$$\begin{aligned} t &= c_2 v_2 + w \\ &= c_1(v_1 + w_2) + w \\ &= c_1 v_1 + c_1 w_2 + w, \end{aligned}$$

since $c_2 = c_1$.
Since W is a subspace, $c_1 w_2 \in W$, and $c_1 w_2 + w \in W$.
So $t = c_1 v_1 + c_1 w_2 + w \in c_1 v_1 + W$.
So $c_2 v_2 + W \subseteq c_1 v_1 + W$.

So $c_1 v_1 + W = c_2 v_2 + W$.

So the operation is well defined.

c) By the associativity of addition in V and the definition of the operation in V/W,

$$\begin{aligned} \big((v_1 + W) + (v_2 + W)\big) + (v_3 + W) &= \big((v_1 + v_2) + W\big) + (v_3 + W) \\ &= \big((v_1 + v_2) + v_3\big) + W \\ &= \big(v_1 + (v_2 + v_3)\big) + W \\ &= (v_1 + W) + \big((v_2 + v_3) + W\big) \\ &= (v_1 + W) + \big((v_2 + W) + (v_3 + W)\big) \end{aligned}$$

for all $v_1 + W, v_2 + W, v_3 + W \in V/W$.

d) By the commutativity of addition in V and the definition of the operation in V/W,

$$\begin{aligned} (v_1 + W) + (v_2 + W) &= (v_1 + v_2) + W \\ &= (v_2 + v_1) + W \\ &= (v_2 + W) + (v_1 + W). \end{aligned}$$

for all $v_1 + W, v_2 + W \in V/W$.

e) The coset $W = 0 + W$ is the zero in V/W since

$$\begin{aligned} W + (v + W) &= (0 + v) + W \\ &= v + W \\ &= (v + 0) + W \\ &= (v + W) + W \end{aligned}$$

for all $v + W \in V/W$.

f) Given any coset $v + W$, its additive inverse is $(-v) + W$ since

$$\begin{aligned} (v + W) + (-v + W) &= v + (-v) + W \\ &= 0 + W \\ &= W \\ &= (-v + v) + W \\ &= (-v + W) + v + W \end{aligned}$$

for all $v + W \in V/W$.

g) Assume $c_1, c_2 \in F$ and $v + W \in V/W$.
Then, by definition of the operation,

$$\begin{aligned} c_1\big(c_2(v + W)\big) &= c_1(c_2 v + W) \\ &= c_1(c_2 v) + W \\ &= (c_1 c_2)v + W \\ &= (c_1 c_2)(v + W). \end{aligned}$$

h) Assume $v + W \in V/W$.
Then, by definition of the operation,

$$\begin{aligned} 1(v + W) &= (1v) + W \\ &= v + W. \end{aligned}$$

i) Assume $c \in F$ and $v_1 + W, v_2 + W \in V/W$.
Then

$$\begin{aligned} c\big((v_1 + W) + (v_2 + W)\big) &= c\big((v_1 + v_2) + W\big) \\ &= c(v_1 + v_2) + W \\ &= (cv_1 + cv_2) + W \\ &= (cv_1 + W) + (cv_2 + W) \\ &= c(v_1 + W) + c(v_2 + W). \end{aligned}$$

j) Assume $c_1, c_2 \in F$ and $v + W \in V/W$.
Then

$$\begin{aligned} (c_1 + c_2)(v + W) &= \big((c_1 + c_2)v\big) + W \\ &= (c_1 v + c_2 v) + W \\ &= (c_1 v + W) + (c_2 v + W) \\ &= c_1(v + W) + c_2(v + W). \end{aligned}$$

So V/W is a vector space over F.

$\Longleftarrow$: Assume W is a subgroup of V and V/W is a vector space over F with action given by

$c(v+W) = cv + W$.
To show: W is a subspace of V.
To show: If $c \in F$ and $w \in W$ then $cw \in W$.
First we show: If $w \in W$ then $w + W = W$.
To show: a) $w + W \subseteq W$.
b) $W \subseteq w + W$.

a) Let $k \in w + W$.
So $k = w + w_1$ for some $w_1 \in W$.
Since W is a subgroup, $w + w_1 \in W$.
So $w + W \subseteq W$.

b) Let $k \in W$.
Since $k - w \in W$, $k = w + (k - w) \in w + W$.
So $W \subseteq w + W$.

Now assume $c \in F$ and $w \in W$.
Then, by definition of the operation on V/W,

$$\begin{aligned} cw + W &= c(w + W) \\ &= c(0 + W) \\ &= c \cdot 0 + W \\ &= 0 + W \\ &= W. \end{aligned}$$

So $cw = cw + 0 \in W$.
So W is a subspace of V. □

(3.2.8) Proposition. *Let $T: V \to W$ be a linear transformation. Let 0_V and 0_W be the zeros for V and W respectively. Then*
a) $T(0_V) = 0_W$.
b) For any $v \in V$, $T(-v) = -T(v)$.

Proof.
a) Add $-T(0_V)$ to both sides of the following equation.

$$T(0_V) = T(0_V + 0_V) = T(0_V) + T(0_V).$$

b) Since $T(v) + T(-v) = T\big(v + (-v)\big) = T(0_V) = 0_W$ and
$T(-v) + T(v) = T\big((-v) + v\big) + T(0_V) = 0_W$, then

$$-T(v) = T(-v). \quad \square$$

(3.2.10) Proposition. *Let $T: V \to W$ be a linear transformation. Then*
a) $\ker T$ is a subspace of V.
b) $\operatorname{im} T$ is a subspace of W.

Proof.
a) By condition a) in the definition of linear transformation, T is a group homomorphism.
By Proposition 1.1.13 a), $\ker T$ is a subgroup of V.
To show: If $c \in F$ and $k \in \ker T$ then $ck \in \ker T$.
Assume $c \in F$ and $k \in \ker T$.
Then, by the definition of linear transformation,

$$T(ck) = cT(k) = c \cdot 0 = 0.$$

So $ck \in \ker T$.

So $\ker T$ is a subspace of V.

b) By condition a) in the definition of linear transformation, T is a group homomorphism.
By Proposition 1.1.13 b), $\operatorname{im} T$ is a subgroup of W.
To show: If $c \in F$ and $a \in \operatorname{im} T$ then $ca \in \operatorname{im} T$.
Assume $c \in F$ and $c \in \operatorname{im} T$.
Then $a = T(v)$ for some $v \in V$.
By the definition of linear transformation,

$$ca = cT(v) = T(cv).$$

So $ca \in \operatorname{im} T$.
So $\operatorname{im} T$ is a subspace of W. □

(3.2.11) Proposition. *Let $T: V \to W$ be a linear transformation. Let 0_V be the zero in V. Then*
a) $\ker T = (0_V)$ if and only if T is injective.
b) $\operatorname{im} T = W$ if and only if T is surjective.

Proof.
Let 0_V and 0_W be the zeros in V and W respectively.
a) $\Longrightarrow$: Assume $\ker T = (0_V)$.
To show: If $T(v_1) = T(v_2)$ then $v_1 = v_2$.
Assume $T(v_1) = T(v_2)$.
Then, by the fact that T is a homomorphism,

$$0_W = T(v_1) - T(v_2) = T(v_1 - v_2).$$

So $v_1 - v_2 \in \ker T$.
But $\ker T = (0_V)$.
So $v_1 - v_2 = 0_V$.
So $v_1 = v_2$.
So T is injective.

$\Longleftarrow$: Assume T is injective.
To show: aa) $(0_V) \subseteq \ker T$.
ab) $\ker T \subseteq (0_V)$.
aa) Since $T(0_V) = 0_W$, $0_V \in \ker T$.
So $(0_V) \subseteq \ker T$.
ab) Let $k \in \ker T$.
Then $T(k) = 0_W$.
So $T(k) = T(0_V)$.
Thus, since T is injective, $k = 0_V$.
So $\ker T \subseteq (0_V)$.
So $\ker T = (0_V)$.

b) $\Longrightarrow$: Assume $\operatorname{im} T = W$.
To show: If $w \in W$ then there exists $v \in V$ such that $T(v) = w$.
Assume $w \in W$.
Then $w \in \operatorname{im} T$.
So there is some $v \in V$ such that $T(v) = w$.
So T is surjective.

$\Longleftarrow$: Assume T is surjective.
To show: ba) $\operatorname{im} T \subseteq W$.
bb) $W \subseteq \operatorname{im} T$.
ba) Let $x \in \operatorname{im} T$.
Then $x = T(v)$ for some $v \in V$.

By the definition of T, $T(v) \in W$.
So $x \in W$.
So $\operatorname{im} T \subseteq W$.

bb) Assume $x \in W$.
Since T is surjective there is a v such that $T(v) = x$.
So $x \in \operatorname{im} T$.
So $W \subseteq \operatorname{im} T$.

So $\operatorname{im} T = W$. □

(3.2.12) Theorem.

a) Let $T: V \to W$ be a linear transformation and let $K = \ker T$. Define

$$\begin{array}{rccc} \hat{T}: & V/\ker T & \to & W \\ & v+K & \mapsto & T(v). \end{array}$$

Then $\hat{T}$ is a well defined injective linear transformation.

b) Let $T: V \to W$ be a linear transformation and define

$$\begin{array}{rccc} T': & V & \to & \operatorname{im} T \\ & v & \mapsto & T(v). \end{array}$$

Then T' is a well defined surjective linear transformation.

c) If $T: V \to W$ is a linear transformation, then

$$V/\ker T \simeq \operatorname{im} T$$

where the isomorphism is a vector space isomorphism.

Proof.

a) To show: aa) $\hat{T}$ is well defined.
ab) $\hat{T}$ is injective.
ac) $\hat{T}$ is a linear transformation.

aa) To show: aaa) If $v \in V$ then $\hat{T}(v+K) \in W$.
aab) If $v_1 + K = v_2 + K \in V/K$ then $\hat{T}(v_1+K) = \hat{T}(v_2+K)$.

aaa) Assume $v \in V$.
Then $\hat{T}(v+K) = T(v)$ and $T(v) \in W$, by the definition of $\hat{T}$ and T.

aab) Assume $v_1 + K = v_2 + K$.
Then $v_1 = v_2 + K$, for some $k \in K$.
To show: $\hat{T}(v_1+K) = \hat{T}(v_2+K)$, i.e.,
To show: $T(v_1) = T(v_2)$.
Since $K \in \ker T$, we have $T(k) = 0$ and so

$$T(v_1) = T(v_2+k) = T(v_2) + T(k) = T(v_2).$$

So $\hat{T}(v_1+K) = \hat{T}(v_2+K)$.

So $\hat{T}$ is well defined.

ab) To show: If $\hat{T}(v_1+K) = \hat{T}(v_2+K)$ then $v_1 + K = v_2 + K$.
Assume $\hat{T}(v_1+K) = \hat{T}(v_2+K)$. Then $T(v_1) = T(v_2)$.
So $T(v_1) - T(v_2) = 0$.
So $T(v_1 - v_2) = 0$.
So $v_1 - v_2 \in \ker T$.
So $v_1 - v_2 = k$, for some $k \in \ker T$.
So $v_1 = v_2 + k$, for some $k \in \ker T$.

To show: aba) $v_1 + K \subseteq v_2 + K$.
abb) $v_2 + K \subseteq v_1 + K$.

aba) Let $v \in v_1 + K$. Then $v = v_1 + k_1$, for some $k_1 \in K$.
So $v = v_2 + k + k_1 \in v_2 + K$, since $k + k_1 \in K$.
So $v_1 + K \subseteq v_2 + K$.

abb) Let $v \in v_2 + K$. Then $v = v_2 + k_2$, for some $k_2 \in K$.
So $v = v_1 - k + k_2 \in v_1 + K$ since $-k + k_2 \in K$.
So $v_2 + K \subseteq v_1 + K$.

So $v_1 + K = v_2 + K$.

So $\hat{T}$ is injective.

ac) To show: aca) If $v_1 + K, v_2 + K \in V/K$ then
$\hat{T}(v_1 + K) + \hat{T}(v_2 + K) = \hat{T}\big((v_1 + K) + (v_2 + K)\big)$.
acb) If $c \in F$ and $v + K \in V/K$ then $\hat{T}\big(c(v + K)\big) = c\hat{T}(v + K)$.

aca) Let $v_1 + K, v_2 + K \in V/K$.
Since T is a homomorphism,

$$\begin{aligned}\hat{T}(v_1 + K) + \hat{T}(v_2 + K) &= T(v_1) + T(v_2) \\ &= T(v_1 + v_2) \\ &= \hat{T}\big((v_1 + v_2) + K\big) \\ &= \hat{T}\big((v_1 + K) + (v_2 + K)\big).\end{aligned}$$

acb) Let $c \in F$ and $v + K \in V/K$.
Since T is a homomorphism,

$$\begin{aligned}\hat{T}\big(c(v + K)\big) &= \hat{T}(cv + K) \\ &= T(cv) \\ &= cT(v) \\ &= c\hat{T}(v + K).\end{aligned}$$

So $\hat{T}$ is a linear transformation.

So $\hat{T}$ is a well defined injective linear transformation.

b) To show: ba) T' is well defined.
bb) T' is surjective.
bc) T' is a linear transformation.

ba) and bb) are proved in Ex. 2.2.3 b), Part I.

bc) To show: bca) If $v_1, v_2 \in V$ then $T'(v_1 + v_2) = T'(v_1) + T'(v_2)$.
bcb) If $c \in F$ and $v \in V$ then $T'(cv) = cT'(v)$.

bca) Let $v_1, v_2 \in V$.
Then, since T is a linear transformation,

$$T'(v_1 + v_2) = T(v_1 + v_2) = T(v_1) + T(v_2) = T'(v_1) + T'(v_2).$$

bcb) Let $v_1, v_2 \in V$.
Then, since T is a linear transformation,

$$T'(cv) = T(cv) = cT(v) = cT'(v).$$

So T' is a linear transformation.

So T' is a well defined surjective linear transformation.

c) Let $K = \ker T$.

By a), the function

$$\begin{array}{rccc} \hat{T}\colon & V/K & \to & W \\ & v+K & \mapsto & T(v) \end{array}$$

is a well defined injective linear transformation.
By b), the function

$$\begin{array}{rcccl} \hat{T}'\colon & V/K & \to & \operatorname{im}\hat{T} & \\ & v+K & \mapsto & \hat{T}(v+K) & = T(v) \end{array}$$

is a well defined surjective linear transformation.
To show: ca) $\operatorname{im}\hat{T} = \operatorname{im}T$.
cb) $\hat{T}'$ is injective.

ca) To show: caa) $\operatorname{im}\hat{T} \subseteq \operatorname{im}T$.
cab) $\operatorname{im}T \subseteq \operatorname{im}\hat{T}$.

caa) Let $w \in \operatorname{im}\hat{T}$.
Then there is some $v + K \in V/K$ such that $\hat{T}(v+K) = w$.
Let $v' \in v + K$.
Then $v' = v + k$ for some $k \in K$.
Then, since T is a linear transformation and $T(k) = 0$,

$$\begin{aligned} T(v') &= T(v+k) \\ &= T(v) + T(k) \\ &= T(v) \\ &= \hat{T}(v+k) \\ &= w. \end{aligned}$$

So $w \in \operatorname{im}T$.
So $\operatorname{im}\hat{T} \subseteq \operatorname{im}T$.

cab) Let $w \in \operatorname{im}T$.
Then there is some $v \in V$ such that $T(v) = w$.
So $\hat{T}(v+K) = T(v) = w$.
So $w \in \operatorname{im}\hat{T}$.
So $\operatorname{im}T \subseteq \operatorname{im}\hat{T}$.

So $\operatorname{im}T = \operatorname{im}\hat{T}$.

cb) To show: If $\hat{T}'(v_1 + K) = \hat{T}'(v_2 + K)$ then $v_1 + K = v_2 + K$.
Assume $\hat{T}'(v_1 + K) = \hat{T}'(v_2 + K)$.
Then $\hat{T}(v_1 + K) = \hat{T}(v_2 + K)$.
Then, since $\hat{T}$ is injective, $v_1 + K = v_2 + K$.

So $\hat{T}'$ is injective.

Thus we have

$$\begin{array}{rccc} \hat{T}'\colon & V/K & \to & \operatorname{im}\hat{T} \\ & v+K & \mapsto & T(v) \end{array}$$

is a well defined bijective linear transformation. □