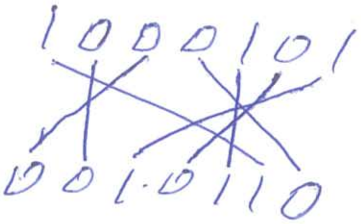
joint work with Persi Diaconis and Andrew Lin

ART Seminar (1)
21.10.2025
A. Ram

The Burnside process

The symmetric group S_n

acts on binary n-tuples (subsets $S \subseteq \{1,\ldots,n\}$)



or

$$\{1,5,7\} \uparrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 7 & 6 & 5 & 1 & 4 \end{pmatrix} \{3,5,6\}$$

Let

$\mathfrak{S}(n) = \{\text{subsets of } \{1,\ldots,n\}\}$ and

$$\mathfrak{S}(n) = \bigsqcup_{k=0}^{n} \mathfrak{S}(n)_k,$$

where $\mathfrak{S}(n)_k = \{S \in \mathfrak{S}(n) \mid |S| = k\}$.

<u>The process:</u> Moving from x to y

Let $x \in \mathfrak{S}(n)$

(a) Choose (uniformly) $w \in S_n$ such that $wx = x$.

(b) Choose (uniformly) $y \in \mathfrak{S}(n)$ such that $wy = y$.

Move from x to y.

The Markov chain is the matrix describing the process.

$K_n(x,y) = $ (probability of moving x to y).

So

$$K_n \in M_{2^n \times 2^n}(\mathbb{R}_{[0,1]}).$$

ART Seminar (2)
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If $n=3$,

$V^{\otimes 3}$ has $\mathbb{C}$-basis $\{000, 100, 010, 001, 110, 101, 011, 111\}$

$\mathbb{C}S_3$ has $\mathbb{C}$-basis $\{123, 213, 132, 321, 231, 312\}$

$$\frac{1}{3!}\begin{pmatrix} 1&1&1&1&1&1 \\ 3&0&3&0&0&0 \\ 3&0&0&3&0&0 \\ 3&3&0&0&0&0 \\ 3&3&0&0&0&0 \\ 3&0&0&3&0&0 \\ 3&0&3&0&0&0 \\ 1&1&1&1&1&1 \end{pmatrix} \frac{1}{2^3}\begin{pmatrix} 1&1&1&1&1&1&1&1 \\ 2&0&0&2&2&0&0&2 \\ 2&2&0&0&0&0&2&2 \\ 2&0&2&0&0&2&0&2 \\ 4&0&0&0&0&0&0&4 \\ 4&0&0&0&0&0&0&4 \end{pmatrix}$$

$$= \frac{1}{24}\begin{pmatrix} 5&1&1&1&1&1&1&5 \\ 3&3&1&1&1&1&3&3 \\ 3&1&3&1&1&3&1&3 \\ 3&1&1&3&3&1&1&3 \\ 3&1&1&3&3&1&1&3 \\ 3&1&3&1&1&3&1&3 \\ 3&3&1&1&1&1&3&3 \\ 5&1&1&1&1&1&1&5 \end{pmatrix}$$

<u>Step 0</u> Notice symmetries of K_n.

$wK_n = K_n w$ and $\overline{K_n} = K_n$ for $w \in S_n$ and $\overline{0}=1$ and $\overline{1}=0$.

<u>Conjecture/now Theorem</u> (Diaconis-Lam-R)

The eigenvalues are

0 with multiplicity 2^{n-1} and

$\beta_{2k} = \frac{1}{2^{2k}}\binom{2k}{k}^2$ with multiplicity $\binom{n}{2k}$

for $k \in \{0, 1, \ldots, \lfloor \frac{n}{2} \rfloor\}$.

ARt Seminar 21.10.2015 (3)

A. Ram

Step 1: (Diaconis–Zhong)

Eigenvalues and eigenvectors of the lumped chain.
Run K_n but only report $|S|$.

$$K_3^{\text{lumped}} = \frac{1}{2^4}\begin{pmatrix} 5 & 3 & 3 & 5 \\ 3 & 5 & 5 & 3 \\ 3 & 5 & 5 & 3 \\ 5 & 3 & 3 & 5 \end{pmatrix}$$ has eigenvectors

$\lvert S\rvert$	$f^{(0)}$	$f^{(1)}$	$f^{(2)}$	$f^{(3)}$
0	1	1	1	1
1	1	$\frac{1}{3}$	-1	-3
2	1	$-\frac{1}{3}$	-1	3
3	1	-1	1	-1
β_i	1	0	$\frac{1}{4}$	0

The eigenvectors are given by Chebyshev polynomials (discrete, 2nd kind)

Step 2 How does K_n relate to K_{n-1}?

$$K_n(1^{\otimes(n-1)} \otimes K_1) = K_{n-1} \otimes K_1$$

If $n=3$ then

$$1^{\otimes 2}\otimes K_1 = \frac{1}{2}\begin{pmatrix} 1&0&0&0&1&0&0&0 \\ 0&1&0&0&0&1&0&0 \\ 0&0&1&0&0&0&1&0 \\ 0&0&0&1&0&0&0&1 \\ 1&0&0&0&1&0&0&0 \\ 0&1&0&0&0&1&0&0 \\ 0&0&1&0&0&0&1&0 \\ 0&0&0&1&0&0&0&1 \end{pmatrix}$$ and $K_3(1^{\otimes 2}\otimes K_1)$ equals

$$\frac{1}{2^6}\begin{pmatrix} 12&4&4&12&12&4&4&12 \\ 8&8&8&8&8&8&8&8 \\ 8&8&8&8&8&8&8&8 \\ 12&4&4&12&12&4&4&12 \\ 12&4&4&12&12&4&4&12 \\ 8&8&8&8&8&8&8&8 \\ 8&8&8&8&8&8&8&8 \\ 12&4&4&12&12&4&4&12 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} K_2 & K_2 \\ K_2 & K_2 \end{pmatrix}$$

since $K_2 = \frac{1}{2^3}\begin{pmatrix} 3&1&1&3 \\ 2&2&2&2 \\ 2&2&2&2 \\ 3&1&1&3 \end{pmatrix}$

<u>Step 3</u> Lifted eigenvectors f_S.

$$f_S(T) = (-1)^{|S\cap T|}\binom{|S|}{|S\cap T|}$$

$$\begin{array}{c|cccccccc}
 & f_{000} & f_{100} & f_{010} & f_{001} & f_{110} & f_{010} & f_{011} & f_{111} \\
\hline
000 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
100 & 1 & -1 & 1 & 1 & -2 & -2 & 1 & -3 \\
010 & 1 & 1 & -1 & 1 & -2 & 1 & -2 & -3 \\
001 & 1 & 1 & 1 & -1 & 1 & -2 & -2 & -3 \\
110 & 1 & -1 & -1 & 1 & 1 & -2 & -2 & +3 \\
101 & 1 & -1 & 1 & -1 & -2 & 1 & -2 & 3 \\
011 & 1 & 1 & -1 & -1 & -2 & -2 & 1 & 3 \\
111 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\
\hline
 & 1 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0
\end{array}$$

eigenvalues

<u>Inner product</u> Define

$$\langle f, g\rangle_\pi = \sum_{S\in S(n)} f(S)\,g(S)\,\pi(S)$$

where $\pi: S(n) \to \mathbb{R}_{[0,1]}$ is the stationary dist.

i.e. $\langle v_S, v_T\rangle = \delta_{ST}\,\frac{1}{n+1}\cdot\frac{1}{\binom{n}{|S|}}$

The lifted eigenvectors are not orthogonal

Step 4 Projected eigenvectors $f_Q^{(\ell)}$

Use Schur-Weyl duality

$$V^{\otimes n} \cong \bigoplus_{\lambda} L(\lambda)\otimes S_n^{\lambda}$$

and project the f_Q onto the components to get new, orthogonal, eigenvectors $f_Q^{(\ell)}$

The eigenvalue pattern for $n=5$:

135 / 24	134 / 25	125 / 34	124 / 35	1345 / 2	1245 / 3	1235 / 4	1234 / 5	12345	$Q \backslash \ell$
								1	0
				0	0	0	0	0	1
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	2
0	0	0	0	0	0	0	0	0	3
				$9/64$	$9/64$	$9/64$	$9/64$	$9/64$	4
								0	5