What is a representation?

A $\mathbb{C}$-algebra is a $\mathbb{C}$-vector space $A$ with a multiplication $\begin{matrix} A\times A \to A \\ (a_1,a_2)\mapsto a_1a_2 \end{matrix}$ which is linear in each factor, is associative and has identity.

Examples $A = M_{n\times n}(\mathbb{C})$ and $A = \mathbb{C}[x]$.

A representation of $A$ is an $A$-module.

An $A$-module is a $\mathbb{C}$-vector space $M$ with an $A$-action $\begin{matrix} A\times M \to M \\ (a,m)\mapsto am \end{matrix}$ which is linear in each factor, is associative and respects identity.

Examples (a) $A = M_{5\times 5}(\mathbb{C})$ and $M = \mathbb{C}^5$ with

$$\begin{matrix} A\times M \to M \\ (a,v)\mapsto av \end{matrix} \quad \text{(matrices act on column vectors).}$$

(b) $A = \mathbb{C}[x]$ and $M = \mathbb{C}^5$. Let $T \in M_{5\times 5}(\mathbb{C})$ and let $A\times M \to M$ given by

$$(c_0 + c_1x + \cdots + c_\ell x^\ell)v = (c_0 + c_1T + \cdots + c_\ell T^\ell)v$$

($x$ acts by the transformation $T$).

## What about representations of groups?

Let $G$ be a group. The <u>group algebra of $G$</u> is the vector space

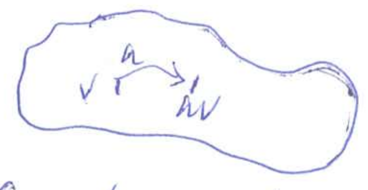
$\mathbb{C}G$ with basis $G$

and multiplication determined by the multiplication on $G$.

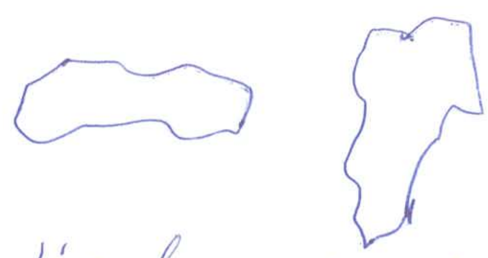
A <u>representation of $G$</u> is a $\mathbb{C}G$-module.

## What does a representation look like?

An <u>irreducible representation of $A$</u> is an $A$-module $M$ that has no submodules except $0$ and $M$.

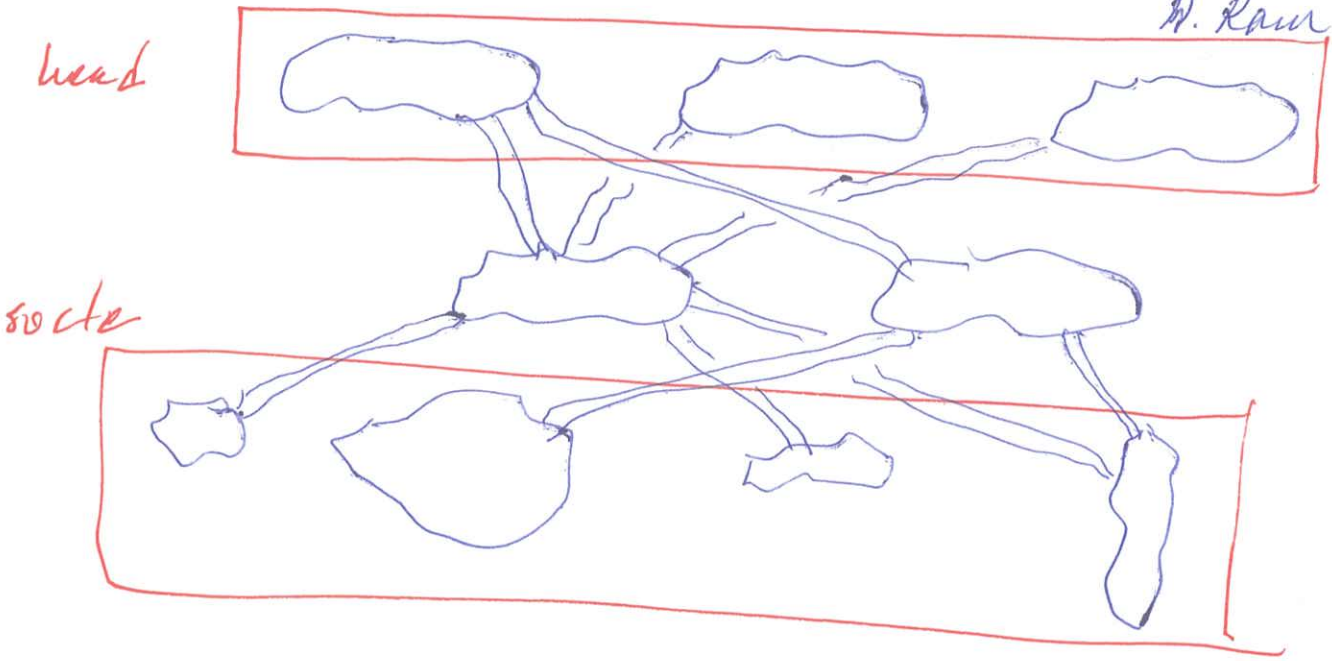



Simple module



direct sum $M \oplus N$ of two simple modules $M$ and $N$.


$0 \to N \hookrightarrow M \twoheadrightarrow P \to 0$.
short exact sequence with $N$ and $P$ simple.




Semisimplification, or take the Grothendieck group, means cut all channels between pools.

The simple modules are the pools $A^\lambda$.

$c \in \mathrm{Ext}^k(A^\lambda, A^\mu)$ means

$$c = (0 \to A^\lambda \to M^{(1)} \to M^{(2)} \to \cdots \to M^{(k-1)} \to A^\mu \to 0)$$

Examples (a) $A = M_n(\mathbb{C})$ has only one simple module $\mathbb{C}^n$.

(b) All simple $\mathbb{C}[x]$-modules are 1-dimensional. These are called eigenvectors of $x$.

if $x$ acts by $\begin{pmatrix} 5 & 1 & & & \\ & 5 & 1 & & \\ & & 5 & & \\ & & & 7 & 1 \\ & & & & 7 \end{pmatrix}$

Where do I find modules?

(1) Any place there is an action

(2)(a) $A$ is an $A$-module.

(b) If $B$ is a subalgebra of $A$ ~~then~~ and
$M$ is an $A$-module then
$\mathrm{Res}_B^A(M)$ - is a $B$module.
If $N$ is a $B$-module then

$$\mathrm{Ind}_B^A(N) = A \otimes_B N = \mathrm{span}\{an \mid a \in A, n \in N\}$$
with $(ab)n = a(bn)$
is an $A$-module

$$\{A\text{modules}\} \underset{\mathrm{Ind}_B^A}{\overset{\mathrm{Res}_B^A}{\rightleftarrows}} \{B\text{-modules}\}$$

(3) Let $M$ be an $A$-module.
The centralizer algebra of $M$ is

$$Z = \mathrm{End}_A(M) = \left\{ \begin{array}{l} \text{linear transfs} \\ z \text{ of } M \end{array} \;\middle|\; \begin{array}{l} \text{If } a \in A \text{ then} \\ azm = zam \end{array} \right\}$$

Then $M$ is a $Z$-module.

13.10.2025. A.L. Bemiaev A. Ram (5)

<u>What do Lie groups have to do with it?</u>

A <u>Lie group</u> is a group $G$ that has a Lie algebra

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\exp} & G \\ \cap\! | & & \cap\! | \\ M_{n\times n}(\mathbb{C}) & \longrightarrow & GL_n(\mathbb{C}) \\ tx & \longmapsto & e^{tx} = g(t) \end{array}$$

If $g_1(t) = e^{tx}$

$g_2(t) = e^{ty}$

then

$$[x,y] = g_1(t)g_2(t)\Big|_{t}$$

(the coefficient of $t$ in $g_1(t)g_2(t)$)

Let $U\mathfrak{g}$ be the algebra generated by $x \in \mathfrak{g}$ with relations

$$xy = yx + [x,y].$$

A <u>representation of $G$</u> is a $U\mathfrak{g}$-module.

Lie groups have favorite subgroups

$$\begin{array}{ccc} GL_n(\mathbb{C}) & \supseteq & G \\ \cup\! | & & \cup\! | \\ B = \left\{\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}\right\} & \supseteq & B \cap G \\ \cup\! | & & \cup\! | \\ T = \left\{\begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}\right\} & \supseteq & T \cap G \end{array}$$

$B \cap G$: Borel subgroup of $G$

$T \cap G$: maximal torus or Cartan subgroup of $G$.