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Formulas for Koornwinder polynomials

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jt with L. Colmenarejo and L. Gagnon

Koornwinder polynomials E_μ for $\mu \in \mathbb{Z}^n$ are
Macdonald polynomials for type CC:
polynomials in $x_1^{\pm 1}, \ldots, x_n^{\pm 1}$
depending on parameters q, t, t_0, u_0, t_n, u_n

For type GL,

(1) Creation formula

(2) Alcove walks

compression $\downarrow$ Around the end

(3) Nonattacking fillings

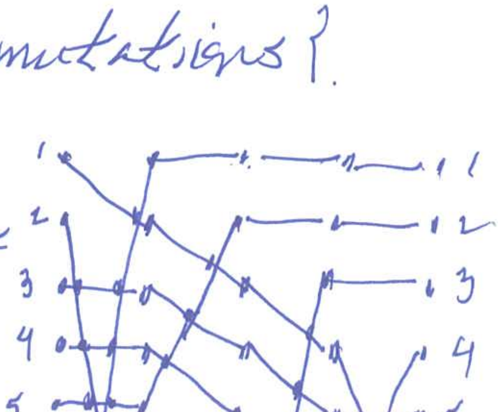
compression $\downarrow$ across a 0 gap

(4) Queue tableaux

$$E_\mu = \sum_T \mathrm{wt}(T)\, x^{\mathrm{end}(T)}$$

Type C: $W_{fin} = \{\text{signed permutations}\}$.

$$z = (\bar{2}\, 5\, \bar{4}\, \bar{1}\, 3) = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} =$$



$$= s_2 s_3 s_4 s_5 s_4 s_3 s_2 s_1 \cdot s_4 s_3 s_2 \cdot s_5 s_4 s_3 \cdot s_4 s_5 s_4$$

Box greedy reduced word

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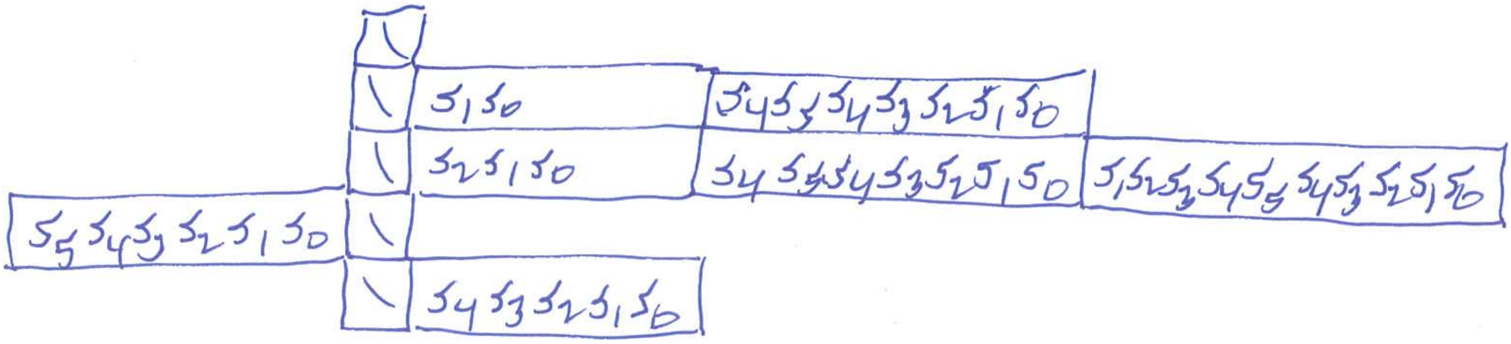
$$\mu = (0,2,3,-1,1) \xrightarrow{s_1 s_0} (-1,0,3,-1,1)$$
$$\xrightarrow{s_2 s_1 s_0} (-2,-1,0,-1,1)$$
$$\xrightarrow{s_4 s_3 s_2 s_1 s_0} (0,-2,-1,0,-1)$$
$$\xrightarrow{s_5 s_4 s_3 s_2 s_1 s_0} (0,0,-2,-1,0)$$
$$\xrightarrow{s_4 s_5 s_4 s_3 s_2 s_1 s_0} (0,0,0,-2,0)$$
$$\xrightarrow{s_4 s_5 s_4 s_3 s_2 s_1 s_0} (-1,0,0,0,0)$$
$$\xrightarrow{s_1 s_2 s_3 s_4 s_5 s_4 s_3 s_2 s_1 s_0} (0,0,0,0,0)$$



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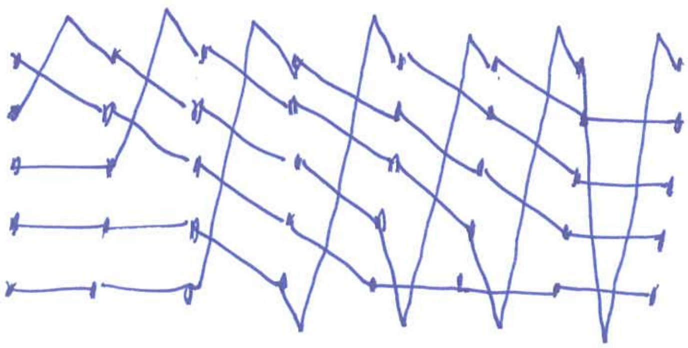

$$= s_{i_1} s_{i_2} \cdots\cdots s_{i_{40}}$$

$$= u_\mu^{\square}$$

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Alcove walks

An *alcove walk* is a choice of factors of $u_\mu^\square$ to cross out.

Each $\cancel{s}_{i_k}$ has:

an inversion: $\beta_k = s_{i_0} \cdots\cdots s_{i_{k+1}} \alpha_{i_k}$

$\alpha_0 = \varepsilon_1 - \frac{1}{2}K$
$\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ and
$\alpha_n = \varepsilon_n$

$s_0 \varepsilon_1 = -\varepsilon_1 + K$
$s_i \varepsilon_i = \varepsilon_{i+1}, \quad s_i \varepsilon_{i+1} = \varepsilon_i$
$s_n \varepsilon_n = -\varepsilon_n$

a permutation $z_k = \bar{s}_{i_1} \cancel{s}_{i_r} \cancel{s}_{i_j} \cdot \bar{s}_{i_{k-1}}$

$\bar{s}_0 = s_1 \cdots s_n \cdots s_1$ and $\bar{s}_i = s_i$ if $i \in \{1, \dots, n\}$.

a weight

$$wt_p(\cancel{s}_{i_k}) = \begin{cases} F^+_{\beta_k}, & \text{if } i_k = 0 \text{ and } z_k(1) < 0, \text{ or} \\ & \text{if } i_k = n \text{ and } z_k(n) > 0, \text{ or} \\ & \text{if } i_k = j \text{ and } z_k(j) < z_k(j+1), \\ F^-_{\beta_k}, & \text{otherwise.} \end{cases}$$

then

$$wt(p) = \prod_{\cancel{s}_{i_k}} wt_p(\cancel{s}_{i_k})$$

$x^{end(p)} = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$ where $(\gamma_1, \dots, \gamma_n) = s_{i_r} \cancel{s}_{i_r} \cancel{s}_{i_j} s_{i_0} (0, \dots, 0)$

and

$$E_\mu = \sum_p wt(p) x^{end(p)}, \text{ sum over alcove walks.}$$

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Compression

A <u>compression sequence</u> S is a sequence of multiple D's passed in derivation of $u_\mu^{\square}$

A <u>compressed alcove walk</u> is an alcove walk p satisfying:

if s_ℓ is crossed out in S

then every $s_j \in S$ before s_ℓ is also crossed out.

Compressed weights

<u>Uncompressed steps:</u> $\mathrm{cwt}_p(s_{i_k}) = \mathrm{wt}_p(s_{i_k})$

<u>Across a D-gap:</u>

$$\mathrm{cwt}_p(\cancel{s}_{d-1}\cdots\cancel{s}_\ell s_{\ell-1}\cdots s_i) = \text{formula}$$

<u>Around the end:</u>

$$\mathrm{cwt}_p(\cancel{s}_k\cdots\cancel{s}_\ell s_{\ell+1}\cdots s_n\cdots s_k) = \text{formula}$$

$$\mathrm{cwt}_p(\cancel{s}_k\cdots\cancel{s}_n\cdots\cancel{s}_\ell s_{\ell-1}\cdots s_k) = \text{formula}$$

Then

$$\mathrm{cwt}(p) = \prod_S \mathrm{cwt}_p(S') \quad \text{product over compression sections}$$

and

$$E_\mu = \sum_p \mathrm{cwt}(p)\, x^{\mathrm{end}(p)}$$

sum over compressed alcove walks.