ART Seminar 1
A. Ram 17.06.2015

$\mathbb{C}$ is algebraically closed

Let $\mathbb{F}$ be a field. For a polynomial

$$p = a_0 + a_1 x + \cdots + a_k x^k \in \mathbb{F}[x]$$

define

$$\begin{array}{rl} f_p : \mathbb{F} & \to \mathbb{F} \\ z & \mapsto ev_z(p), \end{array} \quad \text{where } ev_z(p) = a_0 + a_1 z + \cdots + a_k z^k.$$

A field $\mathbb{F}$ is <u>polysurjective</u> if $\mathbb{F}$ satisfies

if $p \in \mathbb{F}[x]$ and $\deg(p) > 0$ then $f_p : \mathbb{F} \to \mathbb{F}$ is surjective.

<u>Theorem</u> $\mathbb{F}$ is polysurjective if and only if $\mathbb{F}$ is algebraically closed. (The Ax-Grothendieck theorem)

A field $\mathbb{F}$ is <u>eigenvalue rich</u> if $\mathbb{F}$ satisfies

if $n \in \mathbb{Z}_{>0}$ and $A \in M_n(\mathbb{F})$ then there exist $v \in \mathbb{F}^n$ and $\lambda \in \mathbb{F}$ such that $Av = \lambda v$.

<u>Theorem</u> $\mathbb{F}$ is eigenvalue rich if and only if $\mathbb{F}$ is algebraically closed.

A field $\mathbb{F}$ is <u>algebraically closed</u> if $\mathbb{F}$ satisfies

if $p \in \mathbb{F}[x]$ and $\deg(p) > 0$ then there exist $a \in \mathbb{F}$ and $q \in \mathbb{F}[x]$ such that

$$p = (x - a)q.$$

Remove partial fractions from Calculus I

$$\int \frac{17x^5+12x^4-4x^3+7x^2+2x+3}{12x^7-31x^4+15x^2+9x+7}\,dx = \int \frac{p(x)}{q(x)}\,dx.$$

Assume $p(x), q(x) \in \mathbb{F}[x]$ and $\mathbb{F}$ is algebraically closed.

Then

$$\frac{1}{q(x)} = \frac{1}{(x-\lambda_1)^{m_1}\cdots(x-\lambda_\ell)^{m_\ell}}$$ with $\lambda_1,\ldots,\lambda_\ell \in \mathbb{F}$ distinct and $m_1,\ldots,m_\ell \in \mathbb{Z}_{>0}$

Use that, if $\lambda_1 \neq \lambda_2$ then

$$\frac{1}{(x-\lambda_1)^{m_1}(x-\lambda_2)^{m_2}} = \frac{\frac{1}{\lambda_2-\lambda_1}\left((x-\lambda_1)-(x-\lambda_2)\right)}{(x-\lambda_1)^{m_1}(x-\lambda_2)^{m_2}}$$

$$= \frac{\frac{1}{\lambda_2-\lambda_1}}{(x-\lambda_1)^{m_1-1}(x-\lambda_2)^{m_2}} + \frac{\frac{1}{\lambda_1-\lambda_2}}{(x-\lambda_1)^{m_1}(x-\lambda_2)^{m_2-1}}$$

to write

$$\frac{1}{q(x)} = \frac{c_1}{(x-\lambda_1)^{k_1}} + \cdots + \frac{c_r}{(x-\lambda_r)^{k_r}}$$ with $c_1,\ldots,c_r \in \mathbb{F}$, $\lambda_1,\ldots,\lambda_r \in \mathbb{F}$, $k_1,\ldots,k_r \in \mathbb{Z}_{>0}$

Noting that

$$\frac{3x^2+2x+1}{(x-2)^2} = \frac{3(x-2)^2-10(x-2)-21}{(x-2)^2}$$

$$= 3 - \frac{10}{x-2} - \frac{21}{(x-2)^2}$$

write

$$\frac{p(x)}{q(x)} = \frac{c_1 p(x)}{(x-\lambda_1)^{k_1}} + \cdots + \frac{c_r p(x)}{(x-\lambda_r)^{k_r}}$$

$$= a_0 + a_1 x + \cdots + a_n x^n + \frac{b_1}{(x-\lambda_1)^{m_1}} + \cdots + \frac{b_s}{(x-\lambda_s)^{m_s}}$$

with $a_0, \ldots, a_n \in \mathbb{F}$, $b_1, \ldots, b_s \in \mathbb{F}$, $\lambda_1, \ldots, \lambda_s \in \mathbb{F}$
and $m_1, \ldots, m_s \in \mathbb{Z}_{>0}$ and $n \in \mathbb{Z}_{\geq 0}$.

Then do the integral using

$$\int a x^{\ell} dx = \frac{a x^{\ell+1}}{\ell+1} \quad \text{and} \quad \int \frac{b}{(x-\lambda)^m} = \begin{cases} \frac{b}{-m+1}(x-\lambda)^{-m+1}, & \text{if } m>1, \\ b \log(x-\lambda), & \text{if } m=1. \end{cases}$$

ART Seminar (2)
A. Ram 17.06.2015

What is arcsin(x)?

$y = \arcsin(x)$ means $x = \sin(y) = -\frac{1}{2}i(e^{iy} - e^{-iy})$

So $2ixe^{iy} = e^{2iy} - 1$ and $e^{2iy} - 2ixe^{iy} + (ix)^2 - (ix)^2 - 1 = 0$.

So $(e^{iy} - (ix))^2 = 1 - x^2$ and $e^{iy} = \sqrt{1-x^2} + ix$.

So $iy = \log(\sqrt{1-x^2} + ix)$ and

$$\arcsin(x) = y = -i\log(\sqrt{1-x^2} + ix).$$

Why cosh(x) and sinh(x) are so difficult.

With $i^2 = -1$

$$e^{ix} = 1 + ix + \frac{1}{2!}(ix)^2 + \frac{1}{3!}(ix)^3 + \frac{1}{4!}(ix)^4 + \cdots$$

$$= 1 + ix - \frac{1}{2!}x^2 - i\frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$$

$$= \cos(x) + i\sin(x).$$

With $1^2 = 1$,

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$$

$$= \cosh(x) + \sinh(x).$$

Because of this sinh(x) and cosh(x) belong in Calculus 2.

ART Seminar (3)
A. Ram 17.06.2015

What is a^b?

Definition of a^b: $a^b = e^{b\log(a)}$

Theorem If $n \in \mathbb{Z}_{>0}$ then $a^n = \underbrace{a \cdot a \cdots a}_{n \text{ times}}$.

Definition of e^x and $\log(a)$:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots \quad \text{and}$$

$$e^{\log(a)} = a \quad \text{and} \quad \log(e^a) = a.$$

(For which number systems R is e^a (e^x) $\in R$?)
Can you always extend R to be "exponentially closed"?).

Definition of the derivative

Define $\frac{d}{dx} f = (\text{coefficient of } t \text{ in } f(x+t))$

Then

$$\frac{d}{dx} x = (\text{coeff. of } t \text{ in } x+t) = 1.$$

$$\frac{d}{dx}(x^n) = (\text{coeff of } t \text{ in } (x+t)^n) = nx^{n-1} \quad \text{(by the binomial theorem)}$$

$$\frac{d}{dx}(e^x) = (\text{coeff of } t \text{ in } e^{x+t})$$

$$= (\text{coeff of } t \text{ in } e^x e^t) = e^x.$$

Can you prove the product rule and the chain rule with this definition?
How does the proof compare to the proof with an alternate definition?

Things I don't understand

ART Seminar (4)
A. Ram 17.06.2015

(1) Why stationary points instead of critical points?

(2) Concavity: A function $f:(a,b)\to\mathbb{R}$ is concave up on $\mathbb{R}_{(c,d)}$ if for every open interval $I\subseteq\mathbb{R}_{(c,d)}$ every chord of f in I lies above the graph.

i.e. ~~if~~ f is concave up on $\mathbb{R}_{(c,d)}$ if f satisfies:

if $\mathbb{R}_{(\ell,r)}\subseteq\mathbb{R}_{(a,b)}$ and $x\in\mathbb{R}_{(\ell,r)}$ then

$$y=f(\ell)+\frac{f(r)-f(\ell)}{(r-\ell)}(x-\ell)>f(x)$$

What's really at the core here? Curvature

$$\kappa=\left|\frac{d\vec{t}}{ds}\right| \quad \text{curvature}$$

$$\frac{1}{\kappa}=\text{radius of curvature}$$

$$\hat{t}=\frac{f'(t)}{|f'(t)|}=\text{unit tangent vector}$$

$$\hat{n}=\frac{1}{\kappa}\frac{d\hat{t}}{ds}=\text{unit principal normal}$$

$$s(t)=\int_a^t|f'(t)|\,dt=\text{arc length}$$

$$\hat{b}=\hat{t}\times\hat{n}$$

$$\frac{d\hat{b}}{ds}=-\lambda\hat{n} \text{ and } \lambda \text{ is the torsion}$$

Second derivative? Hessian?

Parametrize a curve: $\mathbb{R} \to \mathbb{R}^2$
$t \mapsto (t, f(t))$

Curvature

Let $P \to B$ be a principal G-bundle
ω a connection on P.

The curvature of ω is

$$\Omega = d\omega + \tfrac{1}{2}(\omega \wedge \omega)$$

so that $\Omega(X, Y) = d\omega(X, Y) + \tfrac{1}{2}[\omega(X), \omega(Y)]$
for tangent vectors X, Y to P.

The torsion of ω is

$$\Theta = d\theta + \omega \wedge \theta, \text{ where}$$

θ is the canonical vector valued 1-form on the frame bundle.

The Levi-Civita connection is the unique affine connection on the tangent bundle of a manifold that preserves the Riemannian metric and is torsion free.