03.06.2025
ART Seminar
A. Ram

①

Hyperplanes

$\mathcal{A}_0$ braid arrangement

$\mathcal{A}_0 \cup \mathcal{A}_1$ Shi arrangement

$\mathcal{A}_{\mathbb{Z}} = \bigcup_{\ell \in \mathbb{Z}} \mathcal{A}_\ell$ affine arrangement.

Let $n \in \mathbb{Z}_{>0}$

(a) # regions in $\mathbb{R}^n - \mathcal{A}_0$ is $n!$

(b) # dominant regions in $\mathbb{R}^n - (\mathcal{A}_0 \cup \mathcal{A}_1)$ is

$$C_{1+\frac{1}{n}} = \frac{1}{n+1}\binom{2n}{n} \qquad \text{(Catalan number)}$$

(c) # regions in $\mathbb{R}^n - (\mathcal{A}_0 \cup \mathcal{A}_1)$ is

$$D_{1+\frac{1}{n}} = (n+1)^{n-1}.$$

More generally

$$|W_{fin}| = \prod_{i=1}^{n} d_i, \quad C_{1+\frac{1}{h}} = \prod_{i=1}^{r} \frac{h+1+d_i}{d_i}, \quad D_{1+\frac{1}{h}} = (h+1)^r$$

where $d_1, \ldots, d_r$ are the degrees

$h = d_r$ is the Coxeter number

r is the rank.

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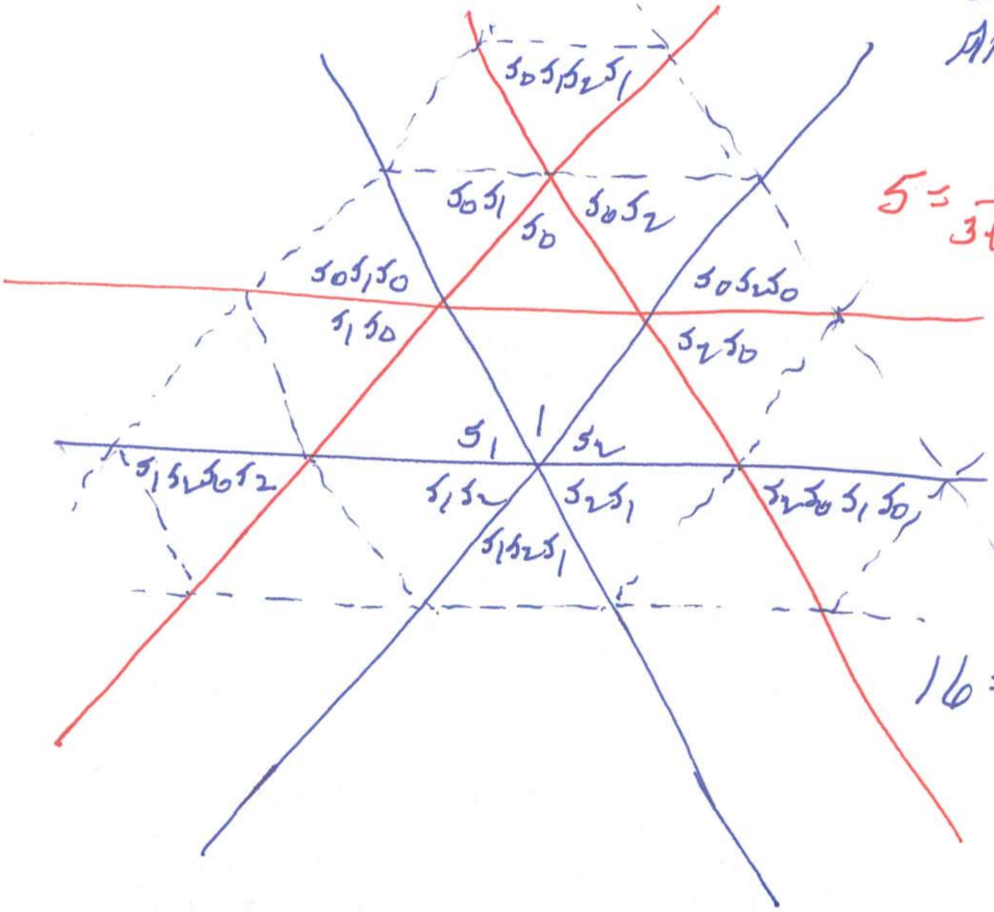
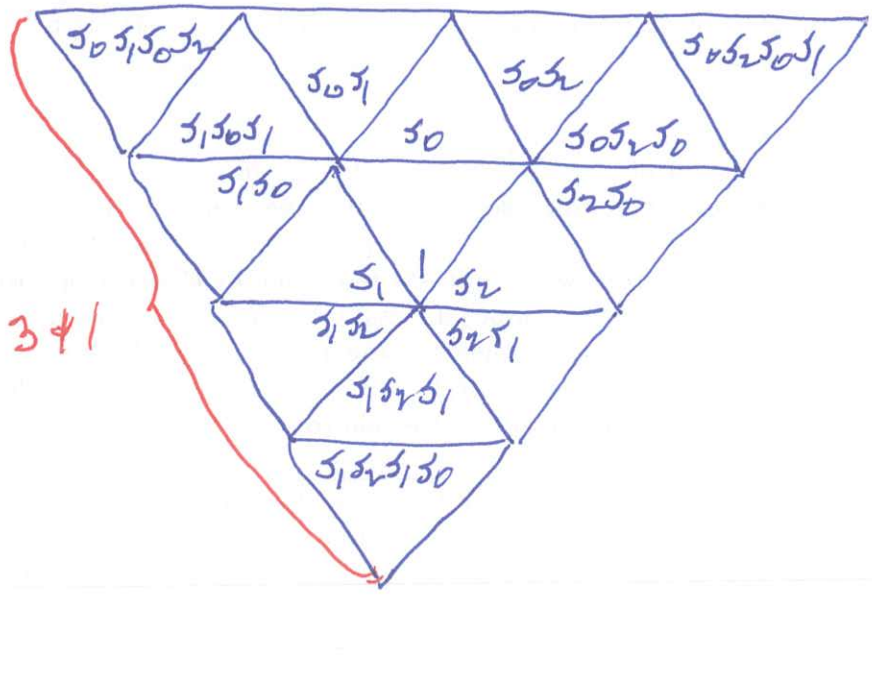


$5 = \frac{1}{3+1}\binom{2\cdot 3}{3}$

$16 = (3+1)^{3-1}$



The affine Weyl group

$\mathfrak{a}_1$ is the fundamental alcove

$s_0, s_1, \dots, s_r$ are reflections in the walls of $\mathfrak{a}_1$

Then
$$w\mathfrak{a}_1 = \mathfrak{a}_w = \mathfrak{a}_{s_{i_1}\cdots s_{i_\ell}}$$
where $s_{i_1}\cdots s_{i_\ell}$ is a minimal length walk from $\mathfrak{a}_1$ to $\mathfrak{a}_w$.

Proposition $$\bigcup_{\text{Shi regions}} \mathfrak{a}_{w^{-1}} = (h+1)\text{-dilate of } \mathfrak{a}_1$$

Let $\mathfrak{b}_w$ be a dominant Shi region and $y^\vee \in \mathfrak{b}_w$

Then
$$\mathfrak{m} = \mathfrak{g}_{\langle y^\vee, \cdot\rangle \geq -1} = \mathfrak{a} \oplus \left(\bigoplus_{\langle y^\vee, \varepsilon_i - \varepsilon_j\rangle \geq -1} \mathbb{C}E_{ij} \right)$$
is a Hessenberg space i.e. a $\mathfrak{b}$-submodule of $\mathfrak{g}$ such that $\mathfrak{m} \supseteq \mathfrak{b}$.

A Dyck path is $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{2n}) \in \{1,-1\}^{2n}$ such that

(a) If $j \in \{1, \dots, 2n\}$ then $\varepsilon_1 + \cdots + \varepsilon_j \leq 0$

(b) $\varepsilon_1 + \cdots + \varepsilon_{2n} = 0$

$$\begin{pmatrix}\text{dominant}\\ \text{Shi}\\ \text{regions}\end{pmatrix} \leftrightarrow \begin{pmatrix}\text{Hessenberg}\\ \text{spaces}\end{pmatrix} \leftrightarrow \begin{pmatrix}\text{Dyck}\\ \text{paths}\end{pmatrix} \leftrightarrow \begin{pmatrix}\text{ad-nilpotent}\\ \text{ideals of } \mathfrak{b}\end{pmatrix}$$

$\mathfrak{b}_w$ $\qquad$ $\mathfrak{m}$ $\qquad$  $\qquad$ $\mathfrak{m}^\perp$

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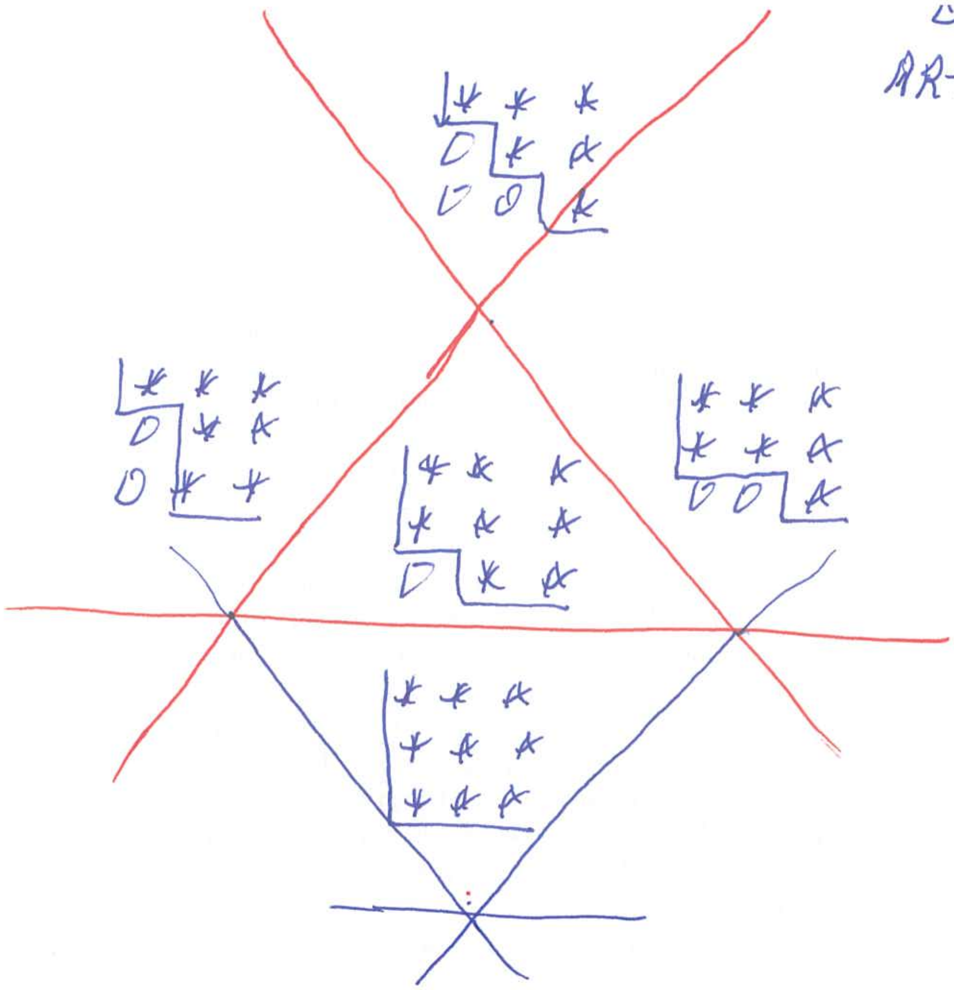
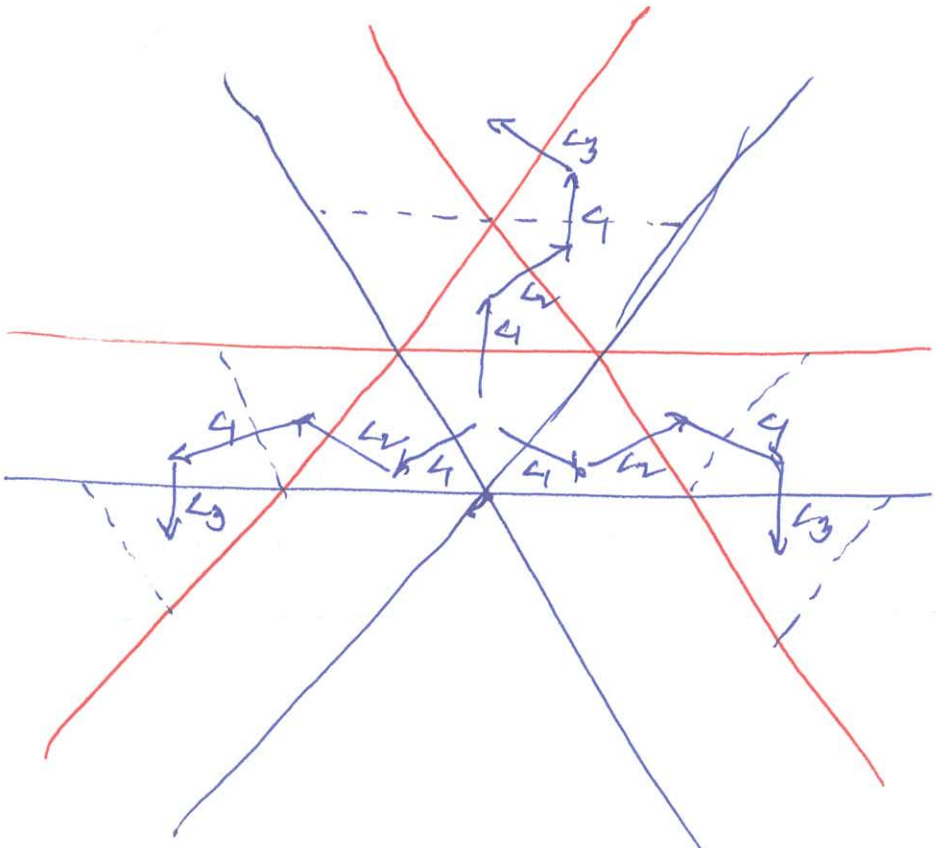
3a



4a



DAHA Modules

H the double affine Hecke algebra

eHe the spherical DAHA.

at

$$q^{h+1}t^{h} = 1.$$

Theorem

(a) eHe has a unique fin. dim. simple module
$eL_{1+\frac{1}{h}}$ of dimension $C_{1+\frac{1}{h}}$

(b) H has a unique fin dim simple module
$L_{1+\frac{1}{h}}$ of dimension $D_{1+\frac{1}{h}}$

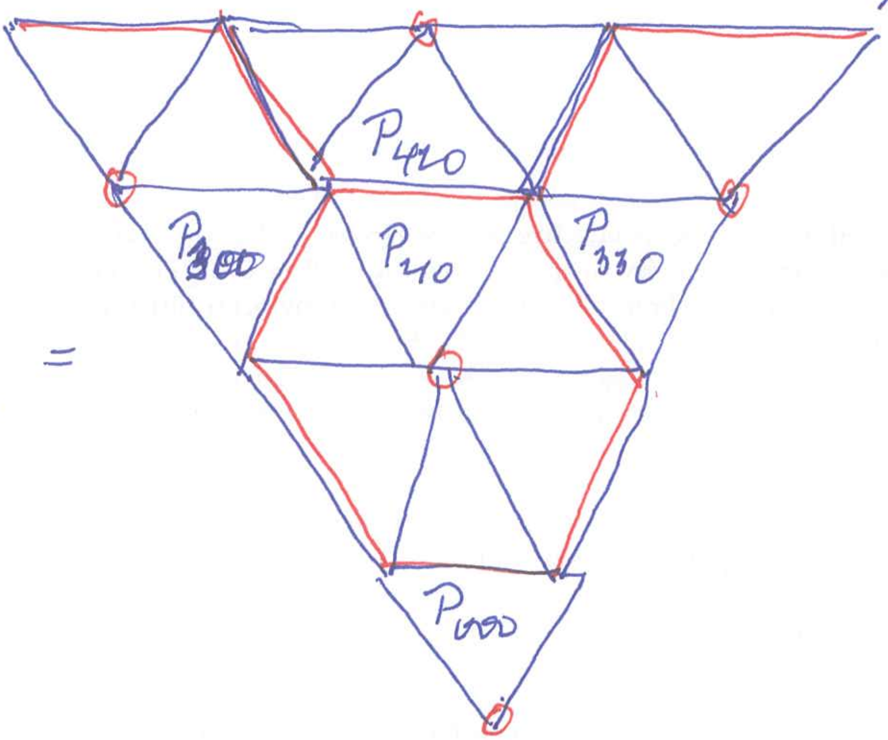
$eL_{1+\frac{1}{h}}$ has basis

$$\{ P_{w \cdot 0+\rho} \mid w \text{ is a dominant Shi region} \}$$

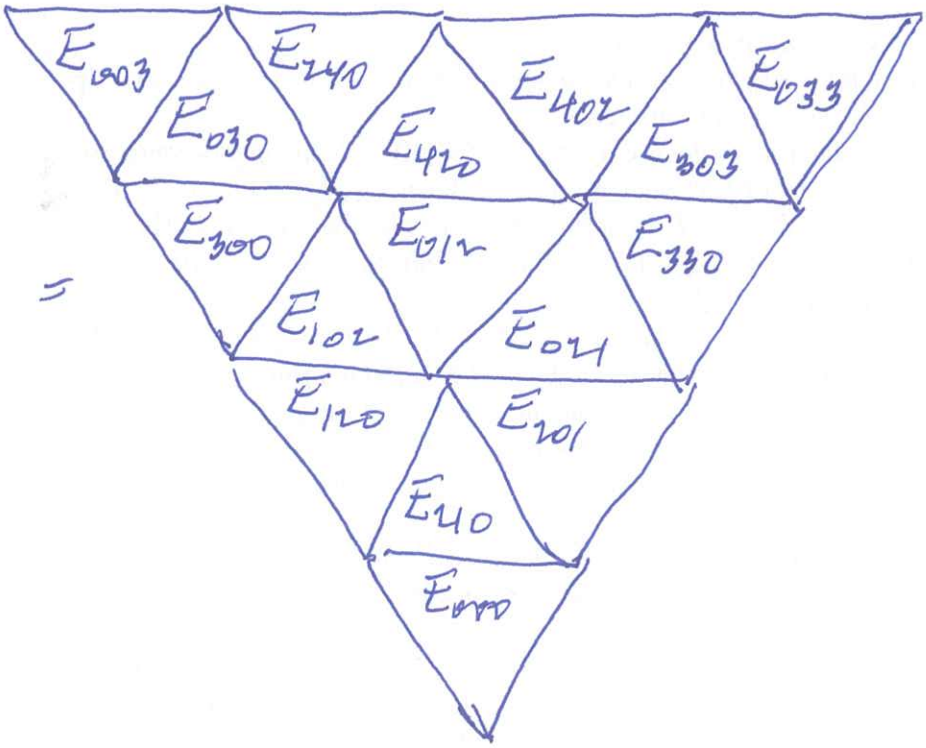
$L_{1+\frac{1}{h}}$ has basis

$$\{ E_{w \cdot 0+\rho} \mid w \text{ is a Shi region} \}.$$

$eL_{1+\frac{1}{n}} =$



$L_{1+\frac{1}{n}} =$



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Affine Springer fibers

$G = GL_n(\mathbb{F}((t)))$ (invertible n-periodic matrices)

$\cup|$

I = Iwahori (upper triangular invertible n-periodic matrices)

$\mathfrak{b} = Lie(I)$ (upper triangular n-periodic matrices)

Let

$$\pi' = \sum_{i \in \mathbb{Z}} E_{i,i+1} \quad \text{and}$$

$$Y_{1+\frac{1}{n}} = \gamma_{\mathfrak{b}}^{-1}(\pi^{-(n+1)}) = \{ gI \in G/I \mid g^{-1}\pi^{-(n+1)} g \in \mathfrak{b} \}.$$

For $i \in \{0,1,\ldots,n-1\}$ and $j \in \mathbb{Z}$, $\hat{E}_{ij} = \sum_{k \in \mathbb{Z}} E_{i+nk,j+nk}$
and $c \in \mathbb{F}$

$$y_i(c) = 1 + (c-1)\hat{E}_{ii} - \hat{E}_{i+1,i+1} + \hat{E}_{i,i+1} + \hat{E}_{i+1,i}$$

Then

$$Y_{1+\frac{1}{n}} = \bigsqcup_{\substack{\text{Shi} \\ \text{regions}}} (Y_{1+\frac{1}{n}} \cap IwI),$$

$$Y_{1+\frac{1}{n}} \cap IwI = IwI \quad \text{unless} \quad w \in \{s_0s_1s_2s_1, s_0s_2s_0s_1, s_0s_1s_0s_2\}$$

$$Y_{1+\frac{1}{n}} \cap I s_0s_1s_2s_1 I = \{ y_0(c_1)y_1(c_2)y_2(c_1)y_1(c_3) \mid c_1,c_2,c_3 \in \mathbb{F} \}$$

$$Y_{1+\frac{1}{n}} \cap I s_0s_2s_0s_1 I = \{ y_0(c_1)y_2(c_2)y_0(c_1)y_1(c_3) \mid c_1,c_2,c_3 \in \mathbb{F} \}$$

$$Y_{1+\frac{1}{n}} \cap I s_0s_1s_0s_2 I = \{ y_0(c_1)y_1(c_2)y_0(c_1)y_2(c_3) \mid c_1,c_2,c_3 \in \mathbb{F} \}.$$