10.11.2011 ①
Hessenberg

Introduction to Hessenberg varieties

Data: $\mathfrak{g}$ semisimple Lie algebra over $\mathbb{C}$

$\cup|$

$\mathfrak{b}$ Borel subalgebra

$\cup|$

$\mathfrak{h}$ Cartan subalgebra

$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ and $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$.

Degrees $n = \dim(\mathfrak{h})$, $W_0 =$ Weyl group

$d_1, \dots, d_n$ are the degrees. $h = d_n =$ Coxeter number

Hessenberg varieties

Let $\mathfrak{m}$ be a $\mathfrak{b}$-submodule of $\mathfrak{g}$.

Let $x \in \mathfrak{g}$

The Hessenberg variety is

$$\mathcal{B}_x^{\mathfrak{m}} = \{gB \in G/B \mid g^{-1}x \in \mathfrak{m}\}$$

Then $\mathcal{B}_x^{\mathfrak{g}} = G/B$ is the flag variety

$\mathcal{B}_x^{\mathfrak{b}}$ is the Springer fiber at x.

$\mathcal{B}_x^{0}$ is empty (unless $x = 0$ when $\mathcal{B}_0^0 = G/B$.

Choices of $x \in \mathfrak{g}$

Nilpotent: $n \in \mathfrak{n}^+$ Semisimple: $s \in \mathfrak{a}$

General: If $x \in \mathfrak{g}$ then there exist unique $s \in \mathfrak{g}$ semisimple, $n \in \mathfrak{g}$ nilpotent with

$$x = s + n \quad \text{and} \quad [s, n] = 0.$$

By conjugation we may assume

$$s \in \mathfrak{a}, \quad n \in \mathfrak{n}^+ \quad \text{and} \quad x = s + n \in \mathfrak{b}$$

Case $\mathfrak{g} = \mathfrak{gl}_n$: Let

$$n_\lambda = \begin{pmatrix} \boxed{\begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{matrix}}\,\lambda_1 & & \\ & \ddots & \\ & & \overset{\lambda_\ell}{\boxed{\begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{matrix}}} \end{pmatrix} \qquad s = \begin{pmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_n \end{pmatrix}$$

$$\left\{ \begin{matrix} G\text{-orbits of} \\ \text{nilpotents} \\ n \in \mathfrak{g} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{partitions } \lambda = (\lambda_1, \ldots, \lambda_\ell) \\ \lambda_1 \geq \cdots \geq \lambda_\ell \\ \lambda_1 + \cdots + \lambda_\ell = n \end{matrix} \right\}$$

$$n_\lambda \longleftarrow\!\!\shortmid \ \lambda$$

$$\left\{ \begin{matrix} G\text{-orbits of} \\ \text{semisimple} \\ s \in \mathfrak{g} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{monic polynomials} \\ \text{of degree } n \end{matrix} \right\}$$

$$s \longmapsto \det(t - s)$$

10.11.2022
Hessenberg (3)

Choices of $\mathfrak{m}$: Writing

$$\mathfrak{g} = \mathfrak{n} \oplus \left(\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \right)$$

let

$$R_{\mathfrak{m}} = \{ \alpha \in R \mid \mathfrak{g}_\alpha \subseteq \mathfrak{m} \}.$$

Bijection:

$$\left\{ \begin{array}{c} \text{ad-nilpotent ideals} \\ \mathfrak{m} \subseteq \mathfrak{n}^+ \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Hessenberg spaces} \\ \mathfrak{m}^\perp \supseteq \mathfrak{b} \end{array} \right\}$$

$$\mathfrak{m} \longmapsto \mathfrak{m}^\perp$$

An abelian ideal is $\mathfrak{m} \subseteq \mathfrak{n}^+$ with $[\mathfrak{m}, \mathfrak{m}] = 0$

$$2^n = \mathrm{Card} \left\{ \begin{array}{c} \text{abelian} \\ \text{ideals of } \mathfrak{b} \end{array} \right\}$$

The Catalan number is

$$\prod_{i=1}^{n} \frac{h+d_i}{d_i} = \mathrm{Card} \left\{ \begin{array}{c} W_f\text{-orbits on} \\ Q^\vee/(h+1)Q^\vee \end{array} \right\}$$

$$= \mathrm{Card} \left\{ \begin{array}{c} \text{ad-nilpotent} \\ \text{ideals of } \mathfrak{b} \end{array} \right\}$$

$$= \mathrm{Card} \left\{ \begin{array}{c} \text{Hessenberg} \\ \text{spaces} \end{array} \right\}$$

$$= \mathrm{Card} \left\{ \begin{array}{c} \text{dominant regions} \\ \text{on Shi arrangement} \end{array} \right\}.$$

$\mathfrak{b}$-submodules of $\mathfrak{gl}_n$

$\underline{n=1}$: $\boxed{0}$ $\boxed{*}$

$\underline{n=2}$: $n=1$, $d=2$, $h=2$

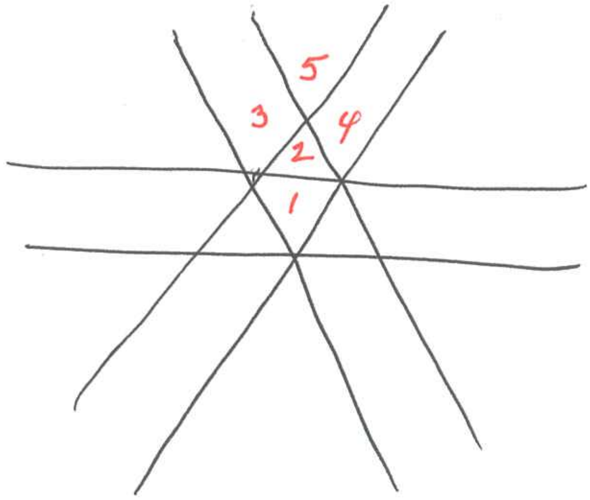
$$\prod_{i=1}^{1} \frac{h+d_i}{d_i} = \frac{2+2}{2} = 2 \quad \text{and} \quad 2^1 = 2.$$

$\underline{n=3}$ $n=2$, $d_1=2$, $d_2=3$, $h=3$

$$\prod_{i=1}^{2} \frac{h+d_i}{d_i} = \frac{(3+2)}{2} \cdot \frac{(3+3)}{3} = 5 \quad \text{and} \quad 2^2 = 4.$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \quad \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & * & * \end{pmatrix} \quad \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \quad \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \quad \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$$



10.11.2022
Hessenberg (5)

$\mathcal{B}_s^{\mathfrak{m}^\perp}$ for s regular semisimple

$\mathfrak{m}^\perp = \mathfrak{g}$: $\quad \mathcal{B}_s^{\mathfrak{g}} = G/B = \bigsqcup_{w \in W} BwB$, where

$$BwB = \{ x_{\beta_1}(c_1) \cdots x_{\beta_\ell}(c_\ell) wB \mid c_1, \dots, c_\ell \in \mathbb{F}_q \}$$

where
$$\{\beta_1, \dots, \beta_\ell\} = \{ \alpha \in R^+ \mid w^{-1}\alpha \in R^- \}$$

$\mathfrak{m}^\perp = \mathfrak{b}$ $\quad \mathcal{B}_s^{\mathfrak{b}} = \{ wB \mid w \in W \}$

$\mathfrak{m} \subseteq \mathfrak{n}^+$ $\quad \mathcal{B}_s^{\mathfrak{m}} = \emptyset$

General $\mathfrak{m}^\perp \supseteq \mathfrak{b}$

$$\left(\mathcal{B}_s^{\mathfrak{m}^\perp} \cap BwB \right) = \{ x_{\beta_1}(c_1) \cdots x_{\beta_\ell}(c_\ell) wB \mid c_1, \dots, c_\ell \in \mathbb{F}_q \}$$

where
$$\{\beta_1, \dots, \beta_\ell\} = \{ \alpha \in R^+ \mid w^{-1}\alpha \in R^- \cap R_{\mathfrak{m}^\perp} \}$$

So
$$\mathrm{Poin}(H^*(\mathcal{B}_s^{\mathfrak{m}^\perp})) = \sum_{w \in W} q^{\ell_{\mathfrak{m}^\perp}(w)}$$

where
$$\ell_{\mathfrak{m}^\perp}(w) = \mathrm{Card}\{ \alpha \in R^+ \mid w^{-1}\alpha \in (R^- \cap R_{\mathfrak{m}^\perp}) \}.$$

Moment graph for $\mathcal{B}_\sigma^{m}$

10.11.2022 (5.5)
Hessenberg

Vertices: $w \in W_0$

Edges: $w \longrightarrow s_\alpha w$ if $\alpha \in R^-_{m}$

So $H^*_T(\mathcal{B}_\sigma^{m}) = \{(f_w)_{w\in W} \mid f_w - f_{s_\alpha w} \in \alpha H_T(pt) \text{ for } \alpha \in R^-_{m}\}$

As an S_n-module

$\mathrm{ch}\, H^*(\mathcal{B}_\sigma^{m}) \otimes \mathrm{sgn})$ = chromatic quasisymmetric function of m.

$$X_{m}(x_1, \ldots x_n; q) = \sum_{\substack{\kappa \text{ proper} \\ \text{colorings}}} q^{\mathrm{asc}(\kappa)} x_1^{\kappa(1)} \cdots x_n^{\kappa(n)}$$

where

$$\mathrm{asc}(\kappa) = \mathrm{Card}\{ \overset{i}{\bullet}\!\!-\!\!\overset{j}{\bullet} \mid i<j \text{ and } \kappa(i) < \kappa(j)\}$$

(see Colmenarejo, Morales, Panova §1.1, arXiv 2104.07599).

The unicellular LLT polynomial is

$$LLT_{m}(x_1, \ldots x_n; q) = \sum_{\substack{\kappa \text{ vertex} \\ \text{colorings}}} q^{\mathrm{asc}(\kappa)} x_1^{\kappa(1)} \cdots x_n^{\kappa(n)}$$

$$= (q-1)^n X_{m}\left(\frac{x}{q-1}; q\right)$$

Paving of $\mathcal{B}_x^{nil}$

$$\mathcal{B}_x^{nil} = \bigsqcup_{w \in W} (\mathcal{B}_x^{nil} \cap BwB)$$

Springer fibers $\mathcal{B}_n^{h}$ with n nilpotent

$\mathcal{B}_{n_\lambda}^{h} \cap BwB$ is empty if $w \notin W^\lambda$

If $w \in W^\lambda$ then

$\mathcal{B}_{n_\lambda}^{h} \cap BwB$ is affine of dimension

As an S_n-module

$$H^*(\mathcal{B}_{n_\lambda}^{h}) \simeq \mathrm{Ind}_{S_\lambda}^{S_n}(\mathrm{sgn}).$$

Note: Since $\mathcal{B}_{n_\lambda}^{h} \subseteq G/B$ then

$$H^*(G/B) \xrightarrow{i^*} H^*(\mathcal{B}_{n_\lambda}^{h})$$

In type A this is surjective.

In general the image is $H^*(\mathcal{B}_{n_\lambda}^{h})^{A(n_\lambda)}$

where $A(n_\lambda) = \dfrac{Z_G(n_\lambda)}{Z_G(n_\lambda)^0}$ is the component group

10.11.2022 (7)
Hessenberg.

Grothendieck-Springer resolutions

$$\mu^{\mathfrak{m}^\perp} : G \times^B \mathfrak{m}^\perp \longrightarrow \mathfrak{g} \qquad \mu^{\mathfrak{m}} : G \times^B \mathfrak{m} \longrightarrow \overline{G\mathfrak{m}}$$
$$(g, m) \longmapsto gm \qquad\qquad (g, m) \longmapsto gm$$

Let
$$N_{\mathfrak{m}} = \dim(G \times^B \mathfrak{m}) = \dim(G/B) + \dim(\mathfrak{m})$$

The <u>Fourier transform</u> relates $\mathfrak{m}$ and $\mathfrak{m}^\perp$

$$\mathcal{F}(R\mu_*^{\mathfrak{m}} \underline{\mathbb{C}}_{\mathfrak{m}}[N_{\mathfrak{m}}]) = R\mu_*^{\mathfrak{m}^\perp}(\underline{\mathbb{C}}_{\mathfrak{m}^\perp}[N_{\mathfrak{m}^\perp}])$$

The stalks are cohomologies:

$$H^{i+N_{\mathfrak{m}^\perp}-\dim\mathfrak{g}}(\mathcal{B}_s^{\mathfrak{m}^\perp}) = R\mu_*^{\mathfrak{m}^\perp}(\underline{\mathbb{C}}_{\mathfrak{m}^\perp}[N_{\mathfrak{m}^\perp}])$$

and if

$$R\mu_*^{\mathfrak{m}}(\underline{\mathbb{C}}_{\mathfrak{m}}[N_{\mathfrak{m}}]) \simeq \bigoplus_{(\mathcal{O},\mathcal{L}) \in \mathcal{O}_{sp}} IC(\mathcal{O}, \mathcal{L}) \otimes V_{\mathcal{O},\mathcal{L}}^{\mathfrak{m}}$$

then

$$R\mu_*^{\mathfrak{m}^\perp}(\underline{\mathbb{C}}_{\mathfrak{m}^\perp}[N_{\mathfrak{m}^\perp}]) \simeq \bigoplus_{\varphi} IC(\mathfrak{g}, \mathcal{M}_\varphi) \otimes V^{\mathfrak{m}}_{\varphi \otimes \mathrm{sgn}}$$

since

$$\mathcal{F}(IC(\mathcal{O},\mathcal{L})) = IC(\mathfrak{g}, \mathcal{M}_{\varphi \otimes \mathrm{sgn}})$$

where $\varphi = \varphi_{\mathcal{O},\mathcal{L}}$ and

$\mathcal{M}_\varphi$ is irred. local system supported on $\mathfrak{g}^{rs}$
corresponding to $\varphi \in \mathrm{Irr}(W)$

Affine Springer fibers

10.11.2022 8

Hessenberg

G = affine Kac-Moody group. I = Iwahori

W = affine Weyl group. $\mathfrak{b} = \mathrm{Lie}(I)$.

Fix $\frac{d}{m} = \nu$ and $w \in W$.

$$\mathbb{G}_m\left(\tfrac{d}{m}\right) = \{ h_\delta(s^m) h_K(s^{-d}) h_{\rho^\vee}(s^d) \mid s \in \mathbb{C}^\times \} \subseteq G$$

$\mathfrak{g}_{d/m}$ = $\frac{d}{m}$-weight space of $\mathfrak{g}$ under $\mathbb{G}_m\left(\frac{d}{m}\right)$-action

$$\mathfrak{b}_w = \mathfrak{g}_{d/m} \cap w\mathfrak{b}w^{-1}$$

$$\mathrm{Lie}(L) = \mathfrak{h} \oplus \left(\bigoplus_{\langle \alpha + r\delta, \frac{d}{m}\rho^\vee \rangle = 0} \mathfrak{g}_{\alpha+r\delta} \right)$$

$$L_w = L \cap wIw^{-1}$$

$$\mathrm{Lie}(P) = \mathfrak{h} \oplus \left(\bigoplus_{\langle \alpha + r\delta, \frac{d}{m}\rho^\vee \rangle \geq 0} \mathfrak{g}_{\alpha+r\delta} \right)$$

Let $\gamma \in \mathfrak{g}_{d/m}$. Then

$S_{P\gamma} \cap PwI$ is an affine space bundle over the Hessenberg

$$\left(L/P_w \right)_\gamma^{\mathfrak{b}_w} = \{ gP_w \in L/P_w \mid g^{-1}\gamma \in \mathfrak{b}_w \}.$$

$\mathfrak{b}$-submodules of $\mathfrak{gl}_n$

n = 1: $\begin{pmatrix} 0 \end{pmatrix}$ $\begin{pmatrix} * \end{pmatrix}$

n = 2: $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ $\begin{pmatrix} * & * \\ * & * \end{pmatrix}$

abelian abelian Hessenberg Hessenberg

$h = 2, \quad d_1 = 2, \quad n = 1$

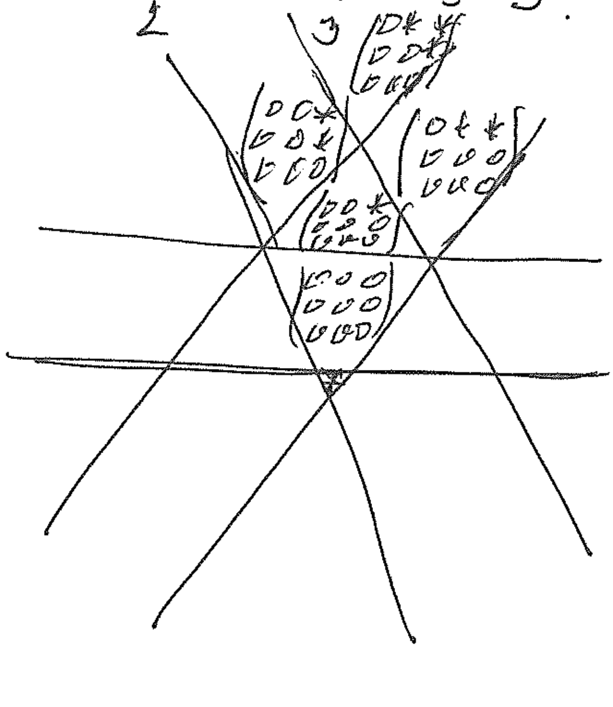
$$\prod_{i=1}^{1} \frac{h+d_i}{d_i} = \frac{2+2}{2} = 2 \quad \text{and} \quad 2! = 2$$

n = 3: $h = 3, \quad d_1 = 2, \quad d_2 = 3, \quad n = 2.$

$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}$

abelian abelian abelian abelian $\leq \mathfrak{n}^+$

$$\prod_{i=1}^{2} \frac{h+d_i}{d_i} = \frac{(3+2)}{2} \cdot \frac{(3+3)}{3} = 5.$$



09.11.2022 ⑥

$H^*(\mathcal{B}_s^{\mathfrak{m}^\perp})$ for s regular semisimple Hessenberg

$\mathfrak{m}^\perp = \mathfrak{g}$: $\mathcal{B}_s^{\mathfrak{g}} = G/B$ and $H^*(G/B)$ has graded character

$$\mathrm{ch}(H^*(G/B)) = \sum_{\lambda \vdash n} t_\lambda s_\lambda = \mathrm{ch}(\mathcal{L}_B^G)$$

where $t_\lambda = K_{\lambda, n}(q) = \dfrac{[n]!}{\prod_{b \in \lambda} [h(b)]}$

$\mathfrak{m}^\perp = \mathfrak{b}$ $\mathcal{B}_s^{\mathfrak{b}} = |W| \cdot pt$ and

$$\mathrm{ch}(H^*(\mathcal{B}_s^{\mathfrak{b}})) = \sum_{\lambda \vdash n} f_\lambda s_\lambda = \mathrm{ch}(\mathbb{C}[W]).$$

$$f_\lambda = K_{\lambda, n}(1) = \frac{n!}{\prod_{b \in \lambda} h(b)} = e$$

General $\mathfrak{m}^\perp$

$\mathrm{ch}(H^*(\mathcal{B}_s^{\mathfrak{m}^\perp})) =$ chromatic symmetric function.

Chromatic quasisym function and LLT polynomials and Modified Macdonald polynomials - Plethysm.

$X_\gamma(x_1,\dots,x_n;t) = \sum_{\kappa \in C(\gamma)} t^{\mathrm{asc}_\gamma(\kappa)} x_{\kappa(1)} \cdots x_{\kappa(n)}$, where

$C(\gamma) = \{\kappa : \{1,\dots,n\} \to \mathbb{Z}_{>0} \mid \text{if } (i,j) \in E(\gamma) \text{ then } \kappa(i) \neq \kappa(j)\}$

and $\mathrm{asc}_\gamma(\kappa) = \#\{(i,j) \in E(\gamma) \mid i<j \text{ and } \kappa(i) < \kappa(j)\}$

Guay-Paquet Hopf algebra / Supercharacters

$\mathrm{Ind}_{UT_\gamma}^{UT_n}(\mathbb{1})$.

$H_T^*(\mathcal{B}_\lambda^m)$

Moment graph

Frobenius Characteristic; Sharashian-Wachs: s reg. ss

Plethysm $\omega(\mathrm{Frob}_q(S(L), H_T^*(S,m), \mathbb{C}[L]) = CSF_q(m)$.

$(t-1)^n X_\gamma\left(\frac{x}{t-1};t\right) = G_\gamma(x;t)$

$\mathrm{Sym} \xrightarrow{P_{KS}^{-1}} \mathrm{cf}_{\mathrm{sup}}^{\mathrm{uni}}(GL_1) \hookrightarrow \mathrm{cf}(GL_q) \xrightarrow{P_1} \mathrm{Sym}$

can be expressed in plethystic notation as

$$f(x) \longrightarrow \omega f\left(\frac{x}{t-1}\right)\Big|_{t=q}.$$