Representations of 2 boundary Hecke and Temperley-Lieb ①

Representation theory seminar 14.08.2018 UniMelb A. Ram

The two boundary Hecke algebra H_k

The two boundary braid group $\mathcal{B}_k$ has generators

$T_0 =$  $T_i =$  $T_k =$  and

relations $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$

$T_0T_1T_0T_1 = T_1T_0T_1T_0$ and $T_kT_{k-1}T_kT_{k-1} = T_{k-1}T_kT_{k-1}T_k$.

Then

$W_j =$  commute in $\mathcal{B}_k$.

The algebra H_k with parameters $t^{\frac{1}{2}}, t_0^{\frac{1}{2}}, t_k^{\frac{1}{2}}$ is $\mathbb{C}\mathcal{B}_k$ with

$$(T_0 - t_0^{\frac{1}{2}})(T_0 + t_0^{-\frac{1}{2}}) = 0,\quad (T_i - t^{\frac{1}{2}})(T_i + t^{-\frac{1}{2}}) = 0\quad (T_k - t_k^{+\frac{1}{2}})(T_k + t_k^{-\frac{1}{2}}) = 0$$

Let H_k^{fin} be the subalgebra of H_k generated by

$$T_1, \dots, T_k.$$

Then

$$H_k = \mathbb{C}[W_1^{\pm1}, \dots, W_k^{\pm1}] \otimes H_k^{fin}$$

$$Z(H_k) = \mathbb{C}[W_1^{\pm1}, \dots, W_k^{\pm1}]^{\mathcal{W}}$$

where $\mathcal{W} = \langle s_1, \dots, s_k \rangle$ acts on $\mathbb{C}[W_1^{\pm1}, \dots, W_k^{\pm1}]$ by

$$\begin{array}{lcl} s_iW_i = W_{i+1} & & s_kW_k = W_k^{-1} \\ s_iW_{i+1} = W_i & \text{and} & s_kW_j = W_j. \\ s_iW_j = W_j & & \end{array}$$

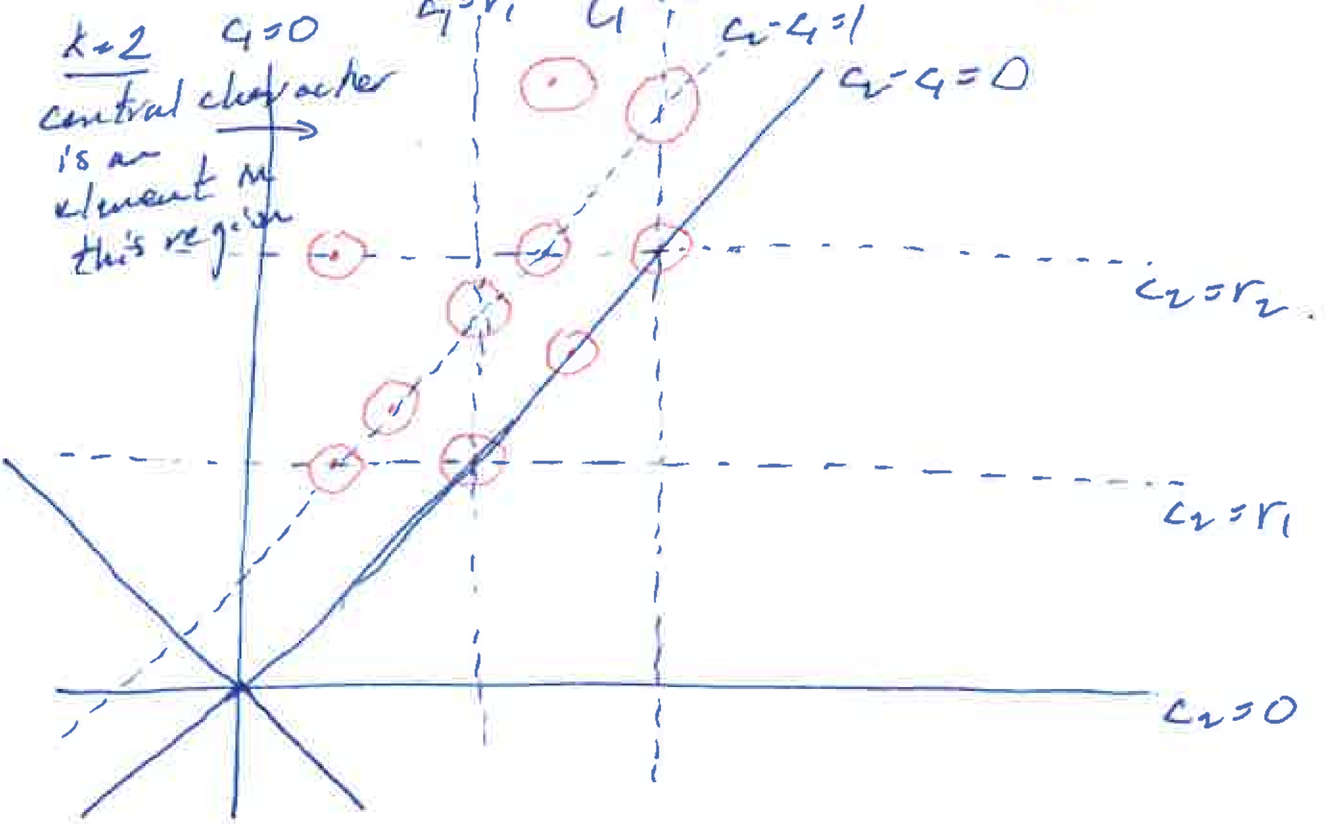
Classifying simple H_k-modules M

(a) $z \in Z(H_k)$ acts on M by a constant.

(b) As $\mathbb{C}[W_1^{\pm 1}, \ldots, W_k^{\pm 1}]$-modules

$$M = \bigoplus_{c \in \mathbb{C}^k} M_c^{gen}, \text{ where } M_c^{gen} = \left\{ m \in M \;\middle|\; \begin{array}{l}\text{There is } \ell \in \mathbb{Z}_{>0} \\ \text{with} \\ (W_i - t^{c_i})^\ell m = 0\end{array} \right\}$$

Let r_1 and r_2 be such that $t^{r_1} = t_0^{-\frac{1}{2}} t_k^{\frac{1}{2}}$ and $t^{r_2} = t_0^{\frac{1}{2}} t_k^{\frac{1}{2}}$



General idea

(a) One simple module for each local region

(b) $\dim(M) =$ size of local region

There are some exceptions.

Exceptions occur only on the boundary of the chamber and on a dotted hyperplane.

<u>Two boundary Temperley-Lieb algebra TL_k</u> A.Ram

HA_2 generated by T_i, T_{i+1} has a unique p_i with

$$p_i^2 = p_i, \quad T_i p_i = -t^{-\frac{1}{2}} p_i, \quad T_{i+1} p_i = -t^{-\frac{1}{2}} p_i.$$

$HB_{2,0}$ generated by T_0, T_1 has unique $p_0, \hat{p}_0$ with

$$p_0^2 = p_0, \quad T_1 p_0 = -t^{-\frac{1}{2}} p_0, \quad T_0 p_0 = -t_0^{-\frac{1}{2}} p_0$$
$$\hat{p}_0^2 = \hat{p}_0, \quad T_1 \hat{p}_0 = -t^{-\frac{1}{2}} p_0, \quad T_0 \hat{p}_0 = +t_0^{\frac{1}{2}} \hat{p}_0.$$

(same with 0 replaced by k and T_1 replaced by T_{k-1}).

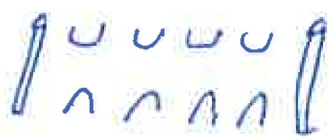
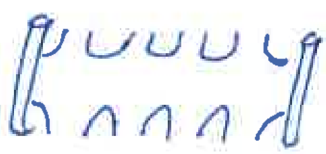
TL_k is the quotient of H_k by

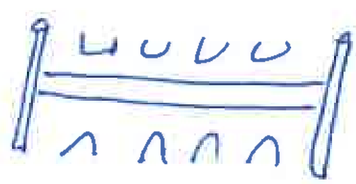
$$p_i = 0, \quad p_0 = 0, \quad \hat{p}_0 = 0, \quad p_k = 0, \quad \hat{p}_k = 0.$$

TL_k has a diagrammatic presentation with

$e_0 = \frac{1}{a_0}(T_0 - t_0^{\frac{1}{2}})$ $\quad e_i = T_i - t^{\frac{1}{2}}$ $\quad e_k = \frac{1}{a_k}(T_k - t_k^{\frac{1}{2}})$

$=$ , $=$  $=$ 

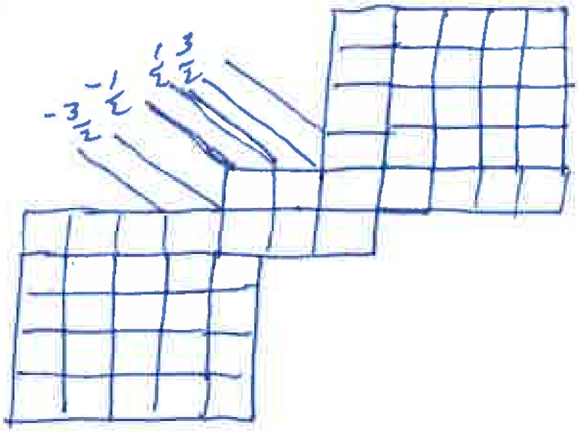
Let $I_1 =$  and $I_2 =$ 

Then $I_1 I_2 I_1 =$ 

TL_k has a basis of <u>noncrossing</u> diagrams

Skew shapes and calibrated H_k-modules

$2k$ boxes



$180°$ rotationally symmetric

A <u>standard tableau of shape λ</u> is a filling of the boxes of λ with $-k, \ldots, -1, 1, \ldots, k$ such that

$\begin{smallmatrix}\boxed{a} & \\ & \boxed{b}\end{smallmatrix}$, $\begin{smallmatrix}\boxed{a} \\ \boxed{b}\end{smallmatrix}$, $\boxed{a \mid b}$ have $a < b$.

<u>and</u> $(-1)(180°\text{rotation of } T) = T$.

$c(T(i))$ = diagonal number of box containing i in T.

Then there is a unique irreducible H_k-module H_k^λ with basis $\{v_T \mid T \text{ is a standard tableau of shape } \lambda\}$

and $W_i v_T = t^{2c(T(i))} v_T$ (calibrated)

and these are <u>all</u> the calibrated irred. H_k-mods.

<u>Theorem</u> H_k^λ is a TL_k-module

$\iff$ λ has ≤ 2 rows.

Standard modules

Method 1: Diagrammatics

Method 2: Induction

Method 3: Geometric.

Method 2: Induction.

Let $J \subseteq \{0, 1, \ldots, k\}$ and

H_J the subalgebra of H_k generated by

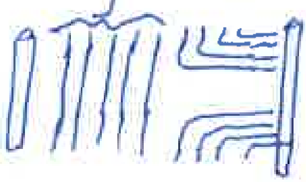
$W_1^{\pm 1}, \ldots, W_k^{\pm 1}$ and T_j for $j \in J$.

Let H_J^μ be a (cuspidal) simple H_J-module.

The induced standard modules are

$$M^{(\mu, J)} = \mathrm{Ind}_{H_J}^{H_k}(H_J^\mu)$$

Method 1: Diagrammatics

$E_j^+ =$  $E_j^- =$ 

The ideals

$$TL_k^{(j)} = \mathbb{C}\text{-span}\left\{ \begin{matrix}\text{diagrams with}\\ \le j \text{ through strands}\end{matrix} \right\}$$

give a filtration

$$TL_k^{(k)} \supseteq TL_k^{(k-1)} \supseteq TL_k^{(k-2)} \supseteq \cdots \supseteq TL_k^{(1)} \supseteq TL_k^{(0)}$$

and if $\prod_{i=1}^{k} (t^i - 1)(t^i - t_0^{\frac{1}{2}} t_k^{\frac{1}{2}})(t^i - t_0^{\frac{1}{2}} t_k^{-\frac{1}{2}}) \neq 0$ then

$$\frac{TL_k^{(j)}}{TL_k^{(j-1)}} \cong End_{\mathbb{C}}(W_+^{(j)}) \oplus End_{\mathbb{C}}(W_-^{(j)})$$

$$TL_k^{(0)} \cong \mathbb{C}[z^{\pm 1}] \otimes End_{\mathbb{C}}(W^{(0)})$$ where

$z =$  and

the diagrammatic standard modules are

$$W_j^{\pm} = \left(\frac{TL_k^{(j)}}{TL_k^{(j-1)}}\right) \cdot E_j^{\pm} \quad \text{and} \quad W^{(0)}(b) = \left(\frac{TL_k^{(0)}}{z=b}\right) \cdot E_0.$$

<u>Method 3: Geometric</u> (Exotic Nilpotent cone).

$$\begin{array}{lcl} G = Sp_{2n}(\mathbb{C}) & & \mathfrak{g}^{ex} = \mathfrak{n}_-^{ex} \oplus \mathfrak{h}^{ex} \oplus \mathfrak{n}_+^{ex} \\ \cup\!\shortmid & & \cup\!\shortmid \\ B \text{ Borel subgroup} & \text{and} & \mathfrak{b}^{ex} = \mathfrak{h}^{ex} \oplus \mathfrak{n}_+^{ex} \\ \cup\!\shortmid & & \cup\!\shortmid \\ T \text{ maximal torus} & & \mathfrak{n}_+^{ex} \end{array}$$

as B-modules where $\mathfrak{g}^{ex} = L(\omega_1) \oplus L(\omega_1) \oplus L(\omega_2)$.

$$\begin{tikzcd} & \tilde{\mathcal{N}}^{ex} = G \times_B \mathfrak{n}_+^{ex} \arrow[dl, "p"'] \arrow[dr, "\mu"] & \\ G/B & & \mathcal{N}^{ex} = G\mathfrak{n}_+^{ex} \end{tikzcd}$$

(exotic Springer resolution)

(exotic nilpotent cone)

Let $(s, q_0, q_1, q_2) \in T \times \mathbb{C}^\times \times \mathbb{C}^\times \times \mathbb{C}^\times$ and

$$X = X_0 + X_1 + X_2 \in \mathcal{N}^{ex}$$

with $sX_0 = q_0 X_0$, $s_1 X_1 = q_1 X_1$ and $sX_2 = q_2 X_2$

The generalized <u>exotic Springer fiber</u> is

$$\mathcal{B}^{ex}_{s,X} = \{ gB \in p(\mu^{-1}(X)) \mid sgB = gB \}$$

<u>Theorem</u> (Kato) There is an H_k-action on

$$M^{s,X} = K_{G \times \mathbb{C}^\times \times \mathbb{C}^\times \times \mathbb{C}^\times}(\mathcal{B}^{ex}_{s,X})$$

The $M^{s,X}$ are the <u>geometric standard modules</u>