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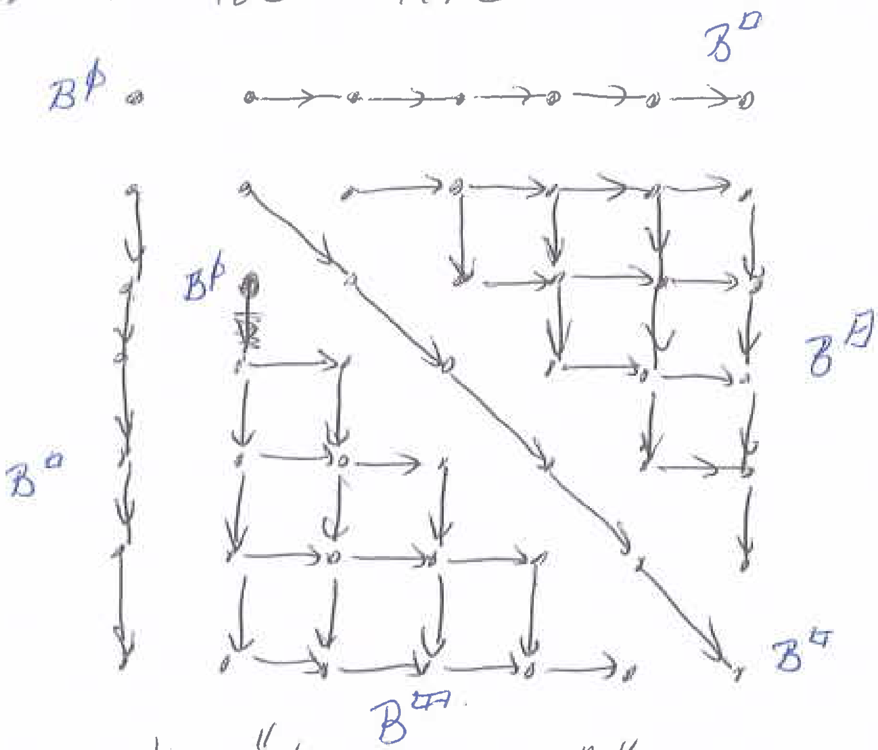
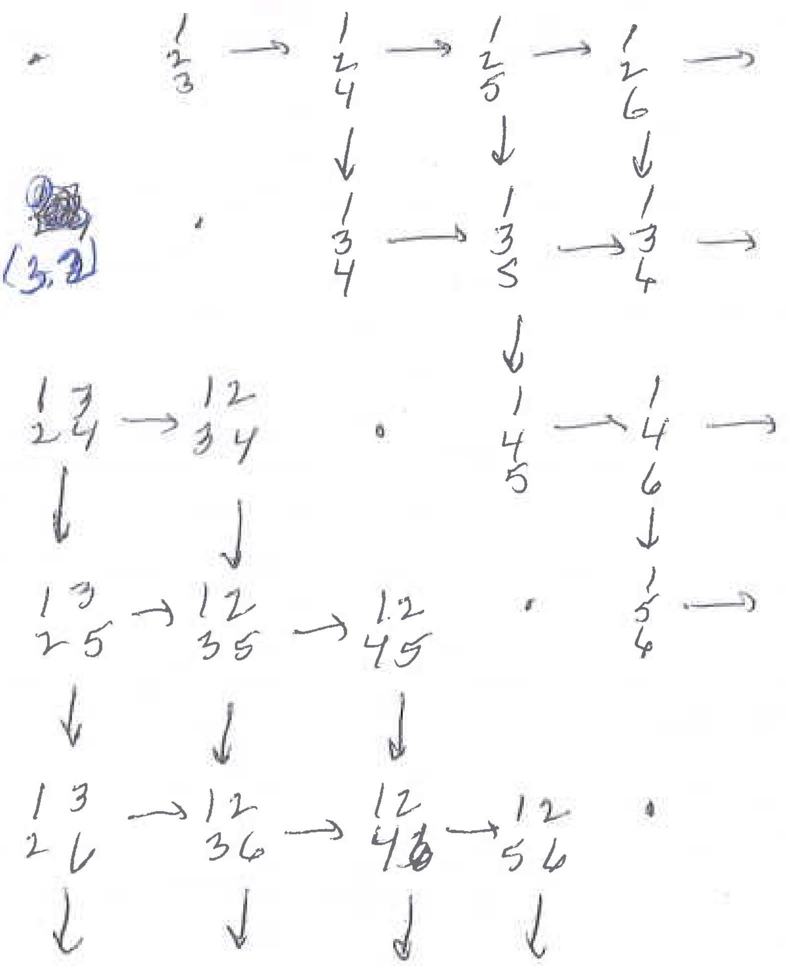
to Toma



Then $B^{\otimes 2}$ looks like




where the "dimension 2" components are




In this picture $B^{\substack{\square\\\square}}$ is identified with $\{(a,b) \mid a<b,\ a\neq 1\}$ as a subset of $(\mathbb{Z}_{>0})^2$, and $B^{\square\square}$ is identified with $\left\{(a,b) \,\middle|\, \begin{array}{l} a>b,\ b\neq 1 \\ \text{and } (a,b)\neq(3,2) \end{array}\right\}$ as a subset of $(\mathbb{Z}_{>0})^2$.

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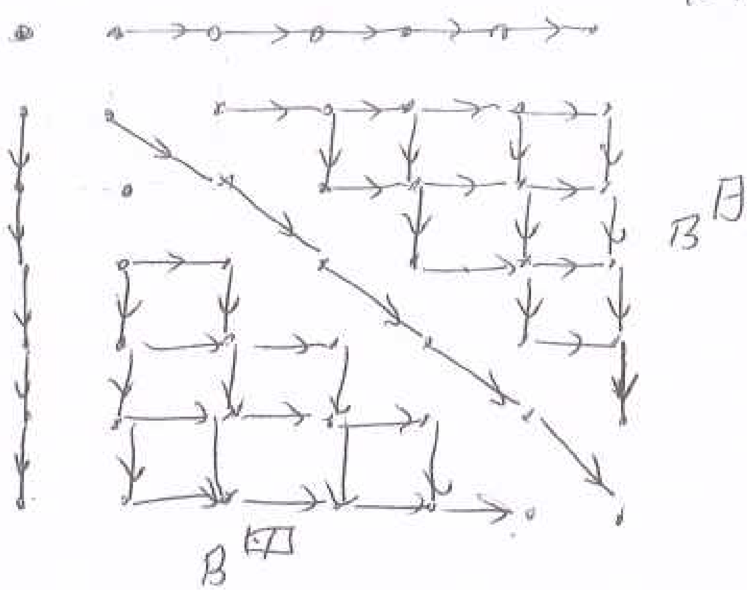
3.5



$B^{\boxminus} = \{ (a,b) \mid a < b,\ a \neq 1 \}$

$B^{\boxplus} = \{ (a,b) \mid a > b,\ b \neq 1,\ (a,b) \neq (3,2) \}$

$B^{\square} = \{ (a,a) \mid a \neq 1 \}$.

$B^{\phi} = \{ (3,2) \}$.  $B^{\phi} = \{ (1,1) \}$.

$B^{\square} = \{ (1,b) \mid b \neq 1 \}$

$B^{\square} = \{ (a,1) \mid a \neq 1 \}$.

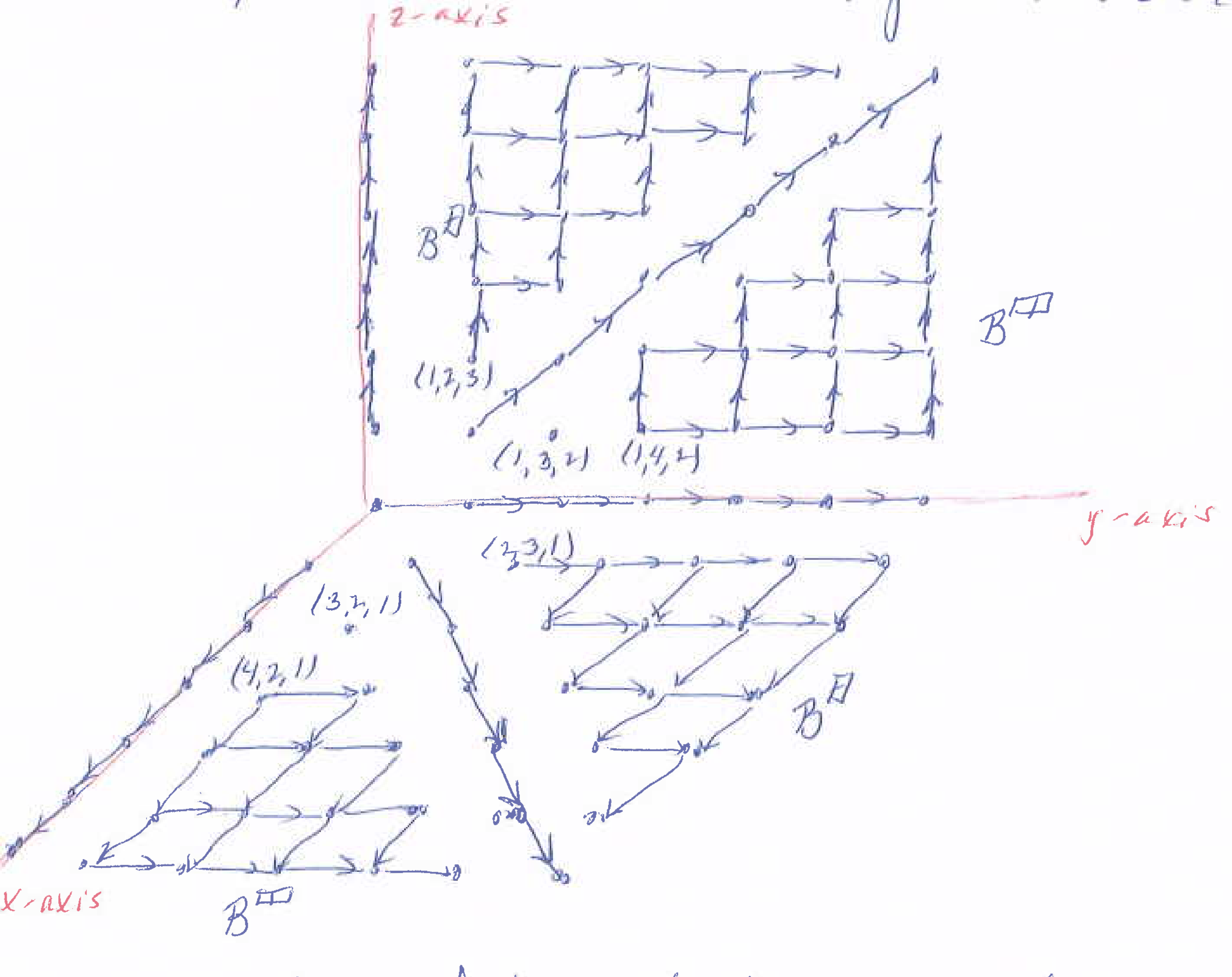
We want to analyze $B^{\otimes 3}$, which will be

$$B^{\otimes 3} = (B\otimes B)\otimes B = B\otimes(B\otimes B)$$

Write elements of $B^{\otimes 3}$ as triples

$(a,b,c)$ with $a,b,c \in \mathbb{Z}_{\geq 1}$

and picture these as sitting in an octant




On each face of the octant we expect to see the same pattern. ($B^{\otimes 2} = 2B^{\emptyset} + 3B^{\square} + B^{\boxminus} + B^{\square\square}$)

We choose the positioning of the components

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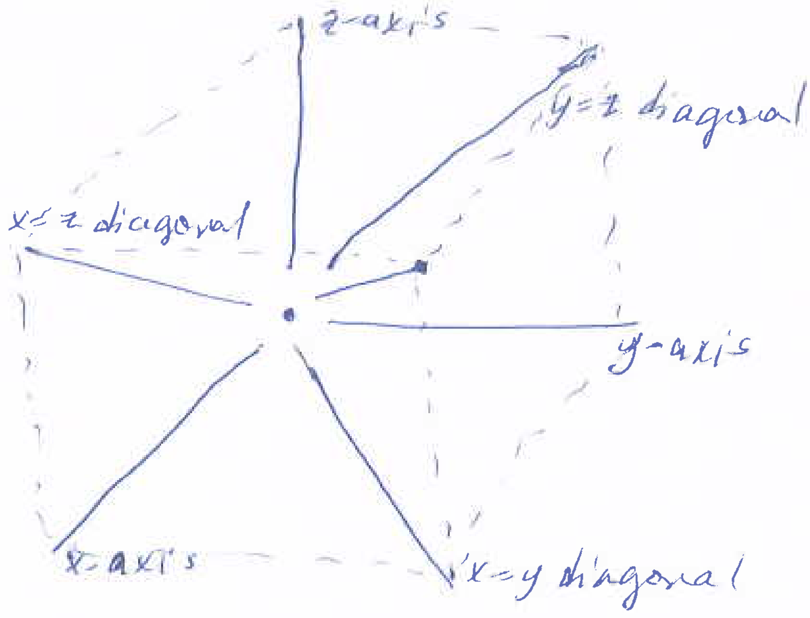


$B^{\Phi} = \{(1,1,1)\}$

$B^{\#} = \{(2,1,1), (3,1,1), (4,1,1), (5,1,1), (6,1,1), (7,1,1)\}$ x-axis.

$B^{\#} = \{(1,2,1), (1,3,1), (1,4,1), (1,5,1), (1,6,1)\ (1,7,1)\}$ y-axis

$B^{\#} = \{(1,1,2), (1,1,3), (1,1,4), (1,1,5), (1,1,6), (1,1,7)\}$ z-axis.

$B^{\#} = \{(2,2,1), (3,3,1), (4,4,1), (5,5,1), (6,6,1), (7,7,1)\}$ x=y diagonal

$B^{\#} = \{(1,2,2), (1,3,3), (1,4,4)\ (1,5,5), (1,6,6), (1,7,7)\}$ y=z diagonal

$B^{\#} = \{(2,1,2), (3,1,3), (4,1,4), (5,1,5), (6,1,6), (7,1,7)\}$ x=z diagonal.

and

$B^{\#} = \{(2,2,2), (3,3,3), (4,4,4)\ (5,5,5), (6,6,6), (7,7,7)\}$ x=y=z diagonal.

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Some dimensions.

$$\dim(B^{\emptyset}) = 1.$$

$$\dim(B^{\square}) = \dim(S_n) = \frac{n(n-1)\cdots 2\cdot 1}{n(n-2)\cdots 2\cdot 1} = n-1$$

$$\dim(B^{\square\square}) = \dim(S_n) = \frac{n(n-1)\cdots 2\cdot 1}{\substack{(n-1)(n-2)(n-4)\cdots 2\cdot 1\\ 2 \quad 1}} = \frac{n(n-3)}{2} = \frac{n^2-3n}{2}$$

$$= \frac{n^2-3n+2}{2} - \frac{2}{2} = \frac{(n-1)(n-2)}{2} - 1.$$

$$\dim(B) = \dim(S_n) = \frac{n(n-1)\cdots 2\cdot 1}{\substack{n(n-3)\cdots 2\cdot 1\\ 2\\ 1}} = \frac{(n-1)(n-2)}{2}$$

$$\dim(B) = \dim(S_n) = \frac{n(n-1)\cdots 2\cdot 1}{\substack{n(n-4)\cdots 2\cdot 1\\ 3\\ 2\\ 1}} = \frac{(n-1)(n-2)(n-3)}{3\cdot 2\cdot 1}$$

$$\dim(S_n) = \frac{n(n-1)\cdots 2\cdot 1}{\substack{(n-2)(n-3)(n-4)(n-6)\cdots 2\cdot 1\\ 3\cdot \ 2 \cdot \ 1}} = \frac{n(n-1)(n-5)}{3\cdot 2\cdot 1}$$

$$= \frac{(n-1)(n^2-5n)}{3\cdot 2\cdot 1} = \frac{(n-1)(n^2-5n+6)}{3\cdot 2\cdot 1} - \frac{6(n-1)}{3\cdot 2\cdot 1}$$

$$= \frac{(n-1)(n-2)(n-3)}{3\cdot 2\cdot 1} - (n-1).$$

$$\dim(S_n) = \frac{n(n-1)\cdots 2\cdot 1}{\substack{(n-1)(n-3)(n-5)\cdots 2\cdot 1\\ 3 \cdot 1}} = \frac{n(n-2)(n-4)}{3}$$

$$= \frac{(n-2)(n^2-4n)}{3} = \frac{(n-2)(n^2-4n+3)}{3} - \frac{(n-2)\cdot 3}{3}$$

$$= \frac{(n-1)(n-2)(n-3)}{3} - ((n-1)-1)$$

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The two dimensional pieces are located on the planes of the faces and on the diagonal planes.

The $yz$-plane $a=1$: $\{(1,b,c) \mid b \neq c\}$, size $(n-1)^2-(n-1)$
$= (n-1)(n-2)$

$B^{\boxminus} \longleftrightarrow \{(1,b,c) \mid b<c\}$, size $\frac{(n-1)(n-2)}{2}$.

$B^{\boxplus} \longleftrightarrow \{(1,b,c) \mid b>c\} - \{(1,3,2)\}$ size $\frac{(n-1)(n-2)}{2} - 1$.

and one extra node $(1,3,2)$

The $xy$-plane $c=1$: $\{(a,b,1) \mid a \neq b\}$, size $(n-1)(n-2)$

$B^{\boxminus} \longleftrightarrow \{(a,b,1) \mid a<b\}$, size $\frac{(n-1)(n-2)}{2}$.

$B^{\boxminus} \longleftrightarrow \{(a,b,1) \mid a>b\} - \{(3,2,1)\}$, size $\frac{(n-1)(n-2)}{2} - 1$

and one extra node $(3,2,1)$.

The $xz$-plane $b=1$: $\{(a,1,c) \mid a \neq c\}$, size $(n-1)(n-2)$

$B^{\boxminus} \longleftrightarrow \{(a,1,c) \mid a<c\}$, size $\frac{(n-1)(n-2)}{2}$

$B^{\boxplus} \longleftrightarrow \{(a,1,c) \mid a>c\} - \{(3,1,2)\}$, size $\frac{(n-1)(n-2)}{2} - 1$

and one extra node $(3,1,2)$.

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The plane $x=y$: $\{(a,a,c) \mid a \neq c\}$ size $(n-1)(n-2)$

$B^{\boxminus} \leftrightarrow \{(a,a,c) \mid a<c\}$, size $\frac{(n-1)(n-2)}{2}$

$B^{\boxplus} \leftrightarrow \{(a,a,c) \mid a>c\} - \{(3,3,2)\}$, size $\frac{(n-1)(n-2)}{2} - 1$.

and one extra node $(3,3,2)$

The plane $x=z$: $\{(a,b,a) \mid a \neq b\}$, size $(n-1)(n-2)$

$B^{\boxminus} \leftrightarrow \{(a,b,a) \mid a<b\}$, size $\frac{(n-1)(n-2)}{2}$

$B^{\boxplus} \leftrightarrow \{(a,b,a) \mid a>b\} - \{(3,2,3)\}$, size $\frac{(n-1)(n-2)}{2} - 1$.

and one extra node $(3,2,3)$

The plane $y=z$: $\{(a,b,b) \mid a \neq b\}$, size $(n-1)(n-2)$

$B^{\boxminus} \leftrightarrow \{(a,b,b) \mid a<b\}$, size $\frac{(n-1)(n-2)}{2}$

$B^{\boxplus} \leftrightarrow \{(a,b,b) \mid a>b\} - \{(3,2,2)\}$, size $\frac{(n-1)(n-2)}{2} - 1$.

and one extra node $(3,2,2)$.

All together these 6 planes account for the $6B^{\boxplus} + 6B^{\boxminus}$ which appear in $B^{\otimes 3}$.

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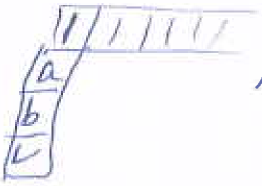
(5)

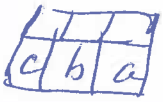
<u>The region</u> $\{(a,b,c) \mid a<b<c\}$, size $\dfrac{(n-1)(n-2)(n-3)}{3\cdot 2\cdot 1}$

$B \longleftrightarrow \{(a,b,c) \mid a<b<c\}$, size $\dfrac{(n-1)(n-2)(n-3)}{3\cdot 2\cdot 1}$


$\longmapsto (a,b,c)$

<u>The region</u> $\{(a,b,c) \mid a>b>c\}$, size $\dfrac{(n-1)(n-2)(n-3)}{3\cdot 2\cdot 1}$

$B \longleftrightarrow \{(a,b,c) \mid a>b>c\} - \left\{ \begin{matrix} (4,3,2), (5,3,2), (6,3,2), (7,3,2), \ldots \\ (5,4,2) \\ (5,4,3) \end{matrix} \right\}$


$\longmapsto (a,b,c)$

size $\dfrac{(n-1)(n-2)(n-3)}{3\cdot 2\cdot 1} - (n-1)$

and $(n-1)$ remaining nodes $(432)\ (532)\ (632)\ (732), \ldots$
$(542)$
$(543)$

<u>The region</u> $\{(a,b,c) \mid a<b>c\}$,

This has size $\dfrac{(n-1)(n-2)(n-3)}{3\cdot 2\cdot 1}\cdot 2 \neq \dfrac{(n-1)(n-2)(n-3)}{3}$

choose 3 from $\{2,3,\ldots,n\}$ and then arrange
$abc$ or $cba$

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to Tan

6

$$B^{\boxplus} \longleftrightarrow \{(a,b,c) \mid a<b>c\} - \left\{ \begin{matrix} 243, 342, \\ 253, 263, 273, \dots \end{matrix} \right\}$$

$$\begin{array}{l} 1\ \alpha \\ a\ c \\ b \end{array} \longmapsto (a,b,c)$$

$$\begin{array}{l} 1\ \alpha \\ a\ b \\ c \end{array} \longmapsto (c,b,a)$$

has size $\dfrac{(n-1)(n-2)(n-3)}{3} - ((n-1)-1)$

and $(n-2)$ remaining nodes $\left\{ \begin{matrix} 243, 342, \\ 253, 263, 273, \dots \end{matrix} \right\}$

The region $\{(a,b,c) \mid a>b<c\}$, size $\dfrac{(n-1)(n-2)(n-3)}{3}$

$$B^{\boxplus} \to \{(a,b,c) \mid a>b<c\} - \left\{ \begin{matrix} 324\ \ 423 \\ 523, 623, 723, \dots \end{matrix} \right\}$$

$$\begin{array}{l} 1\ \alpha \\ b\ a \\ c \end{array} \longmapsto (c,b,a)$$

$$\begin{array}{l} 1\ \alpha \\ b\ c \\ a \end{array} \longmapsto (a,b,c)$$

has size $\dfrac{(n-1)(n-2)(n-3)}{3} - ((n-1)-1)$

and $n-2$ remaining nodes $\left\{ \begin{matrix} 324\ \ 423 \\ 523, 623, 723, \dots \end{matrix} \right\}$

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(7)

These account for all the 3-dimensional components that we need in $B^{\otimes 3}$

$$B^{\yng(1,1,1)} + 2B^{\yng(2,1)} + B^{\yng(3)}$$

The remaining nodes in the 3-dimensional pieces: are

$\{$ 324, 423,
513, 623, 723 . . . $\}$ (n-2 nodes)

$\{$ 243, 342
253, 263, 273, . . . $\}$ (n-2 nodes)

$\{$ 432, 532, 632, 732 . . .
542
543 $\}$ (n-1 nodes)

$\{322\}$

$\{323\}$

$\{332\}$

$\{312\}$

$\{321\}$

$\{132\}$

$\{111\}$

to give $5B^{\phi} + 10B^{\square}$ in combination with the $7B^{\square}$ coming from the list on page 1.