TALK

Are there S_n-crystals?

① Kronecker problem

S_n^λ the irreducible $\mathbb{C}S_n$ module indexed by $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ with $\lambda_i \in \mathbb{Z}_{\geq 0}$, $\lambda_1 + \lambda_2 + \cdots = n$.

Compute $g_{\mu\nu}^\lambda$, where

$$S_n^\mu \otimes S_n^\nu = \bigoplus_\lambda (S_n^\lambda)^{\oplus g_{\mu\nu}^\lambda}.$$

② Littlewood-Richardson rule

$L_n(\lambda)$ the irred. finite dim. $GL_n(\mathbb{C})$-module $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$, with $\lambda_i \in \mathbb{Z}_{\geq 0}$, $\lambda_{n+1} = 0$.

Compute $c_{\mu\nu}^\lambda$, where

$$L_n(\mu) \otimes L_n(\nu) = \bigoplus_\lambda L_n(\lambda)^{\oplus c_{\mu\nu}^\lambda}$$

(also with $GL_n(\mathbb{C})$ replaced by reductive alg. G).

Solution to ②:

$B_n(\lambda)$ the crystal graph of $L_n(\lambda)$

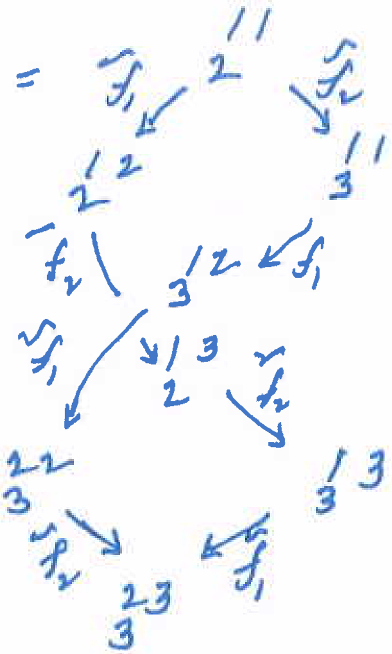
$c_{\mu\nu}^\lambda$ = # of components connected of the labeled graph $B_n(\mu) \otimes B_n(\nu)$ with highest weight λ.

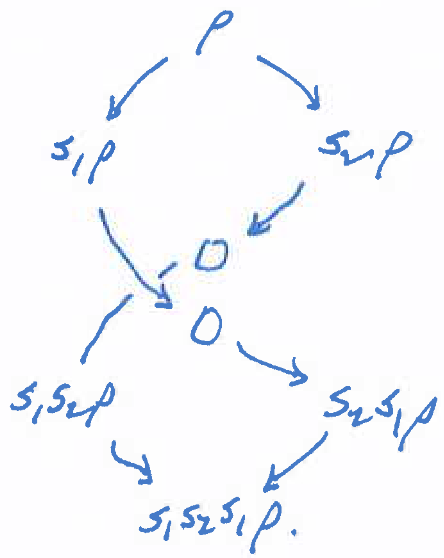
Is there a similar solution for S_n-modules?

Examples

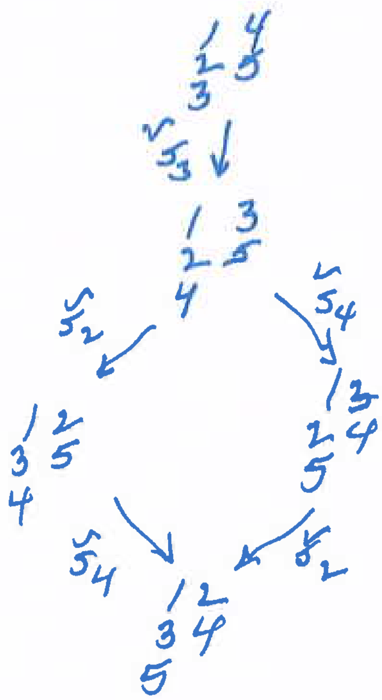
Weights

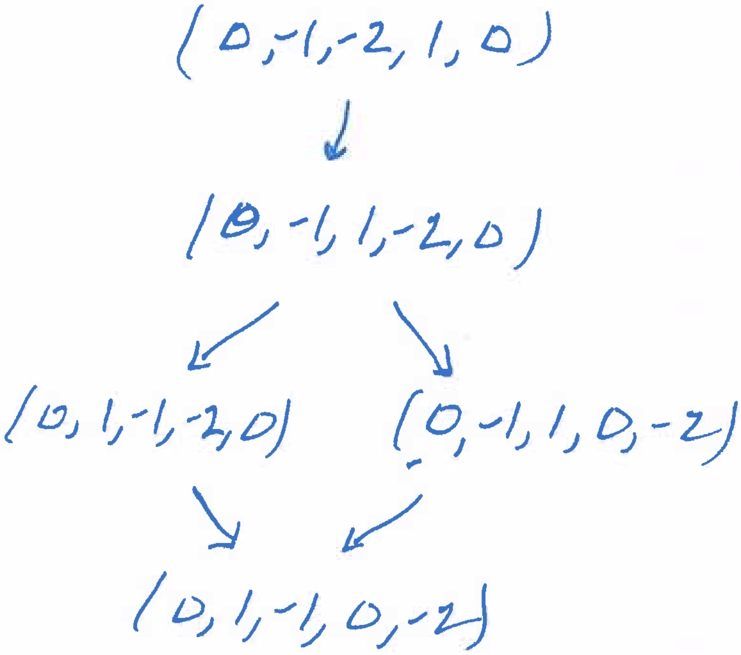
$\mathcal{B}_3(\boxplus) =$





$\mathcal{B}_5^{\boxplus} =$





The <u>highest weight</u> of $\mathcal{B}_n^{\lambda}$ is the column reading tableau.

TALK

Tensor categories – graph categorification ③

We want

$\mathcal{B}_n = \{$certain labeled graphs$\}$

$\oplus$ is disjoint union $\sqcup$

<u>simple objects</u> B_n^λ for $\lambda \vdash n$,

which are connected graphs in $\mathcal{B}_n$

$$\mathrm{Card}(B_n^\lambda) = \dim(S_n^\lambda)$$

<u>monoidal structure</u> $\pi : B_n^\mu \otimes B_n^\nu \longrightarrow \bigsqcup_\lambda (B_n^\lambda)^{\oplus \gamma_{\mu\nu}^\lambda}$

with

$$\pi(\pi(a \otimes b) \otimes c) = \pi(a \otimes \pi(b \otimes c))$$

and <u>Grothendieck ring</u>

$$K(\mathcal{B}_n) \xrightarrow{\sim} K(\mathbb{C}S_n\text{-mod})$$

$$[B_n^\lambda] \longmapsto [S_n^\lambda]$$

<u>Theorem</u> If $n \geq 3$ then $\mathcal{B}_n$ does not exist.

Example $n=3$

$\otimes$	$S_3^{(3)}$	$S_3^{(2,1)}$	$S_3^{(1,1,1)}$
$S_3^{(3)}$	$S_3^{(3)}$	$S_3^{(2,1)}$	$S_3^{(1,1,1)}$
$S_3^{(2,1)}$	$S_3^{(2,1)}$	$S_3^{(3)} \oplus S_3^{(2,1)} \oplus S_3^{(1,1,1)}$	$S_3^{(2,1)}$
$S_3^{(1,1,1)}$	$S_3^{(1,1,1)}$	$S_3^{(2,1)}$	$S_3^{(3)}$

Let $A = 1\,2\,3$ $\quad B = \begin{smallmatrix}1 & 3\\ 2 & \end{smallmatrix}$ $\quad C = \begin{smallmatrix}1 & 2\\ 3 & \end{smallmatrix}$ $\quad D = \begin{smallmatrix}1\\ 2\\ 3\end{smallmatrix}$ and give

	A	B	C	D
A	A	B	C	D
B	B			
C	C			
D	D			A

permutation of (A,B,C,D)

permutation of (B,C)

with

$$(xy)z = x(yz).$$

IMPOSSIBLE.

Stability (Murnaghan, see Brion 1993 §3.4 Cor 1)

For $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ let $\bar\lambda = (\lambda_2 \geq \lambda_3 \geq \cdots)$

For $n > 2(\mu_2 + \nu_2)$

$\gamma^{\lambda}_{\mu\nu}$ depends only on $\bar\mu, \bar\nu, \bar\lambda$.

So look at B_n^λ as n gets large.

Define

$B_\infty^{\bar\lambda}$ to be the stable limit of B_n^λ

$\dim(B_\infty^{\bar\lambda}) = \#$ of boxes of $\bar\lambda$

$\mathrm{dir}(B_\infty^{\bar\lambda}) = \#$ of removable boxes of $\bar\lambda$.

TODO: Make a graphical tensor category B_∞ with simple objects

$B_\infty^{\bar\lambda}$ for $\bar\lambda = (\lambda_2 \geq \lambda_3 \geq \cdots)$ with $\lambda_i \in \mathbb{Z}_{\geq 0}$

(1) Use it to compute $\varphi^{\bar\lambda}_{\bar\mu\bar\nu}$

(2) Define $\Gamma_n : B_\infty \to B_n$

and use these to compute $\gamma^\lambda_{\mu\nu}$.

(see Sam-Snowden, ..., Inna Entova-Aizenbud).

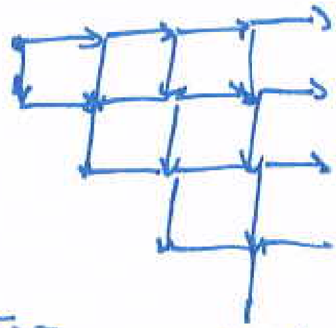
Examples

$$B_n^{(n)} = \{123\cdots n\} \qquad B_\infty^{\emptyset} = \bullet$$

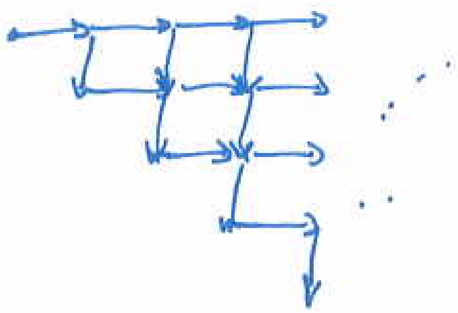
$$B_n^{(n-1,1)} = \left\{ \begin{smallmatrix}1\,3\,4\cdots\\2\end{smallmatrix} \xrightarrow{\check{s}_2} \begin{smallmatrix}1\,2\,4\,5\cdots\\3\end{smallmatrix} \xrightarrow{\check{s}_3} \begin{smallmatrix}1\,2\,3\,5\cdots\\4\end{smallmatrix} \xrightarrow{\check{s}_4} \cdots \right\}$$

$$B_\infty^{\square} = \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \cdots$$

$$B_n^{(n-2,2)} = \begin{tikzcd}
\begin{smallmatrix}1\,3\,5\,6\cdots\\2\,4\end{smallmatrix} \arrow[r] \arrow[d] & \begin{smallmatrix}1\,3\,4\,6\,7\cdots\\2\,5\end{smallmatrix} \arrow[r] \arrow[d] & \begin{smallmatrix}1\,3\,4\,5\,7\cdots\\2\,6\end{smallmatrix} \arrow[r] \arrow[d] & \cdots \\
\begin{smallmatrix}1\,2\,5\,6\cdots\\3\,4\end{smallmatrix} \arrow[r] & \begin{smallmatrix}1\,2\,4\,6\,7\\3\,5\end{smallmatrix} \arrow[r] \arrow[d] & \begin{smallmatrix}1\,2\,4\,5\,7\\3\,6\end{smallmatrix} \arrow[r] \arrow[d] & \cdots \\
 & \begin{smallmatrix}1\,2\,3\,6\,7\\4\,5\end{smallmatrix} \arrow[r] & \begin{smallmatrix}1\,2\,3\,5\,7\\4\,6\end{smallmatrix} \arrow[r] \arrow[d] & \cdots \\
 & & {} &
\end{tikzcd}$$

$B_\infty^{(2,2)} =$  $\cdots$

$$B_n^{(n-2,1,1)} = \begin{tikzcd}
\begin{smallmatrix}1\,4\,5\,6\\2\\3\end{smallmatrix} \arrow[r] & \begin{smallmatrix}1\,3\,5\,6\cdots\\2\\4\end{smallmatrix} \arrow[r] \arrow[d] & \begin{smallmatrix}1\,3\,4\,6\\2\\5\end{smallmatrix} \arrow[r] \arrow[d] & \cdots \\
 & \begin{smallmatrix}1\,2\,5\,6\\3\\4\end{smallmatrix} \arrow[r] & \begin{smallmatrix}1\,2\,4\,6\\3\\5\end{smallmatrix} \arrow[r] \arrow[d] & \cdots \\
 & & \begin{smallmatrix}1\,2\,3\,6\\4\\5\end{smallmatrix} \arrow[r] & \cdots
\end{tikzcd}$$

$B_\infty^{(1,1)} =$ 

Halverson-Lewandorski and Geometry

TALK

① As $GL_n(\mathbb{C}) \times S_k$-modules

$$V^{\otimes k} \cong \bigoplus_{\substack{\lambda \vdash k \\ \lambda_{n+1}=0}} L_n(\lambda) \otimes S_k^{\lambda}, \quad \text{where } V = \mathbb{C}^n.$$

A crystal version of this is

$$\left\{ \begin{matrix} \text{words of length } k \\ \text{from } \{1,2,\dots,n\} \end{matrix} \right\} \xleftrightarrow{RSK} \bigsqcup_{\substack{\lambda \vdash k \\ \lambda_{n+1}=0}} B_n(\lambda) \otimes B_k^{\lambda}$$

$$i_1 i_2 \cdots i_k \longmapsto (P, Q)$$

column strict shape λ

standard shape λ

② As $S_n \times P_k(n)$ modules

$$V^{\otimes k} \cong \bigoplus_{\substack{\lambda \vdash n \\ |\bar{\lambda}| \le k.}} S_k^{\lambda} \otimes P_k(n)^{\lambda}, \quad \text{where } V = \mathbb{C}^n$$

The Halverson-Lewandorski

$$\left\{ \begin{matrix} \text{words of length } k \\ \text{from } \{1,2,\dots,n\} \end{matrix} \right\} \xleftrightarrow{HL} \bigsqcup_{\substack{\lambda \vdash n \\ |\bar{\lambda}| \le k}} B_n^{\lambda} \times C_k(n)^{\lambda}$$

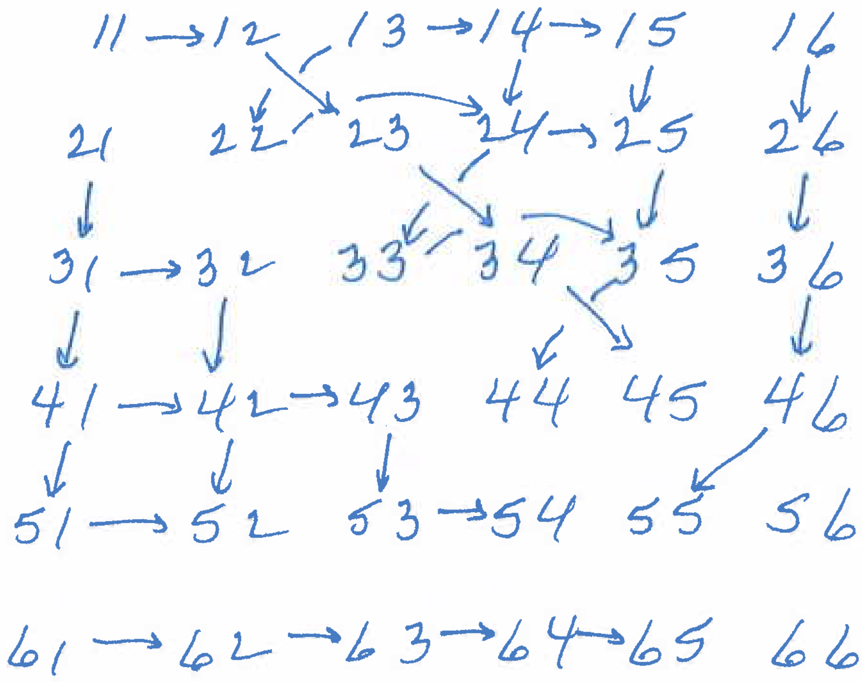
$$i_1 i_2 \cdots i_k \longmapsto (Q, Z)$$

standard shape λ

up down tableau length k ending in λ.

Examples

① Halverson-Lewandowski with $n=2$.



② My "improved" version