Are there S_n-crystals?

Kronecker problem:

S_n^λ the simple $\mathbb{C}S_n$-module

$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ with $\lambda_i \in \mathbb{Z}_{\geq 0}$, $\lambda_1 + \lambda_2 + \cdots = n$

Compute $\gamma_{\mu\nu}^\lambda$ where

$$S_n^\mu \otimes S_n^\nu \cong \bigoplus_\lambda (S_n^\lambda)^{\oplus \gamma_{\mu\nu}^\lambda}$$

Littlewood-Richardson

$L_n(\lambda)$ the simple $GL_n(\mathbb{C})$-module

$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ with $\lambda_i \in \mathbb{Z}_{\geq 0}$ and $\lambda_{n+1} = 0$.

Compute $c_{\mu\nu}^\lambda$ where

$$L_n(\mu) \otimes L_n(\nu) \cong \bigoplus_\lambda L_n(\lambda)^{\oplus c_{\mu\nu}^\lambda}$$

Let $B_n(\lambda)$ be the crystal of $L_n(\lambda)$.

$c_{\mu\nu}^\lambda$ = # of connected components of highest weight λ in the labeled graph $B_n(\mu) \otimes B_n(\nu)$.

Are there B_n^λ, S_n-crystals, which might compute the $\gamma_{\mu\nu}^\lambda$.

Tensor categories - Graphical categorification

We want

$$\mathcal{B}_n = \{\text{certain graphs}\}$$

$\oplus$ is disjoint union $\sqcup$

simple objects $\mathcal{B}_n^\lambda$ for $\lambda \vdash n$
the connected objects in $\mathcal{B}_n$

$$\dim(S_n^\lambda) = \mathrm{Card}(\mathcal{B}_n^\lambda)$$

monoidal structure $\pi : \mathcal{B}_n^\mu \otimes \mathcal{B}_n^\nu \xrightarrow{\sim} \bigsqcup_\lambda (\mathcal{B}_n^\lambda)^{\sqcup g_{\mu\nu}^\lambda}$

with $\pi(\pi(a \otimes b) \otimes c) = \pi(a \otimes \pi(b \otimes c))$

Grothendieck group

$$K(\mathcal{B}_n) \xrightarrow{\sim} K(S_n\text{-mod})$$
$$[\mathcal{B}_n^\lambda] \longmapsto [S_n^\lambda]$$

Theorem $\mathcal{B}_n$ doesn't exist,
except for $n=1$ and $n=2$.

Stability (Murnaghan, see Brion 1993)

For $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ let $\bar\lambda = (\lambda_2 \geq \lambda_3 \geq \cdots)$

For $n > 2(\mu_2 + \nu_2)$,

$\gamma_{\mu\nu}^{\lambda}$ depends only on $\bar\mu, \bar\nu, \bar\lambda$.

So look at B_n^λ as n gets large.

Define

$B_\infty^{\bar\lambda}$ to be the stable limit of B_n^λ

$\dim(B_\infty^{\bar\lambda}) = \#$ of boxes of λ

$\operatorname{div}(B_\infty^{\bar\lambda}) = \#$ of removable boxes of $\bar\lambda$.

TODO: Make a graphical tensor category B_∞ with simple objects $B_\infty^{\bar\lambda}$, $\bar\lambda = (\lambda_2 \geq \lambda_3 \geq \cdots)$

1. Use it to compute $\varphi_{\bar\mu\bar\nu}^{\bar\lambda}$
2. Define $\Gamma_n : B_\infty \to B_n$ and use them to compute $\gamma_{\mu\nu}^{\lambda}$.

(see Sam-Snowden, … ….

and Inna Entova-Aizenbud).

Halverson-Lewandorski and geometry

(1) As $GL_n(\mathbb{C}) \times S_k$-modules

$$V^{\otimes k} \simeq \bigoplus_{\substack{\lambda \vdash k \\ \lambda_{n+1}=0}} L_n(\lambda) \otimes S_k^\lambda, \quad \text{where } V = \mathbb{C}^n.$$

The crystal version of this is

$$\left\{\begin{matrix}\text{words of} \\ \text{length } k \text{ from} \\ \{1,2,\ldots n\}\end{matrix}\right\} \xleftrightarrow{RSK} \bigsqcup_{\substack{\lambda \vdash k \\ \lambda_{n+1}=0}} B_n(\lambda) \times B_k^\lambda$$

$$i_1 i_2 \cdots i_k \longmapsto (P, Q)$$

standard shape λ

column strict shape λ

(2) As $S_n \times P_k(n)$-modules,

$$V^{\otimes k} \simeq \bigoplus_{\substack{\lambda \vdash n \\ |\bar\lambda| \le k}} S_n^\lambda \otimes P_k(n)^\lambda, \quad \text{where } V = \mathbb{C}^n.$$

The crystal? version of this is

$$\left\{\begin{matrix}\text{words of length } k \\ \text{from } \{1,2,\ldots,n\}\end{matrix}\right\} \xleftrightarrow{HL} \bigsqcup_{\substack{\lambda \vdash n \\ |\bar\lambda| \le k}} B_n^\lambda \times C_k(n)^\lambda$$

$$i_1 i_2 \cdots i_k \longmapsto (X, Y)$$

add remove of length k ending in λ.

standard shape λ