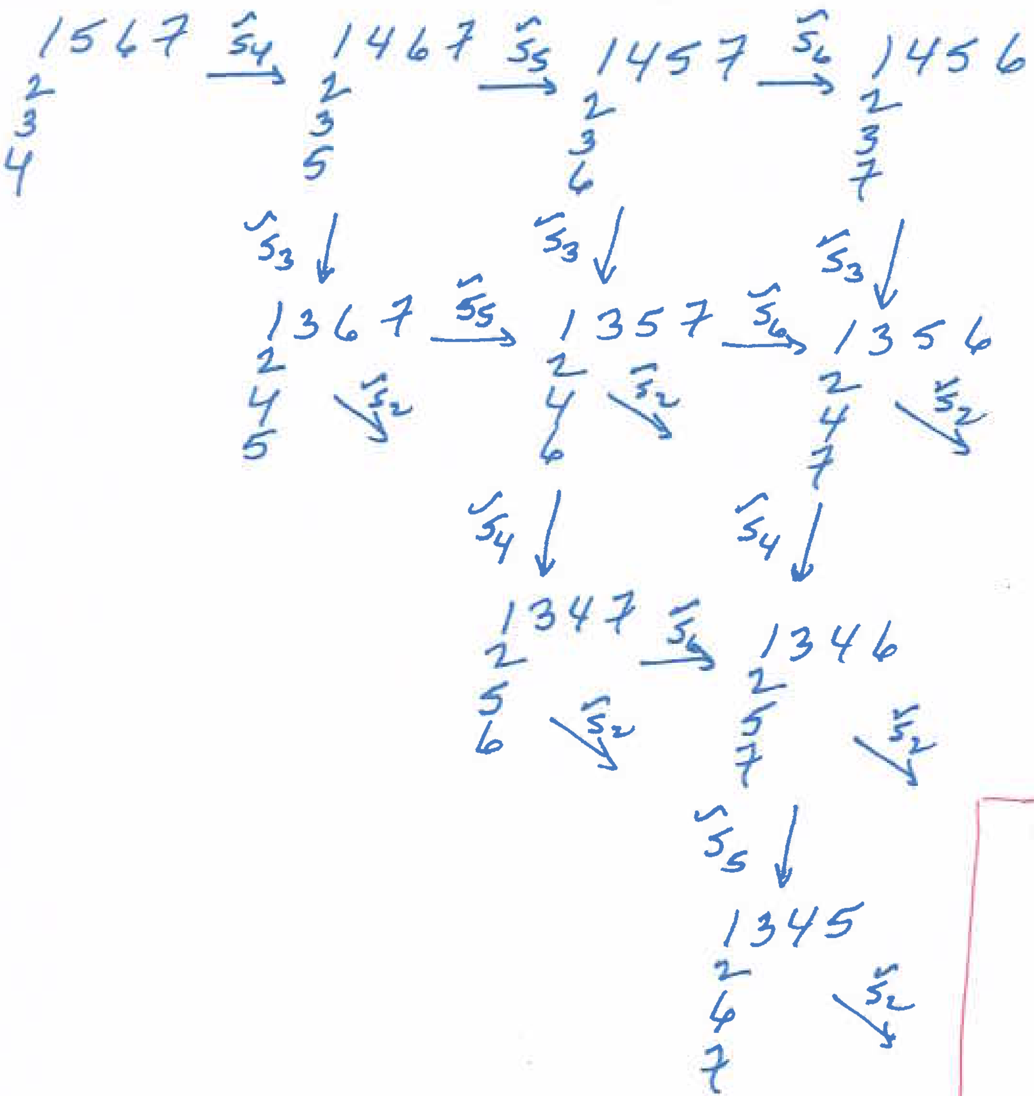
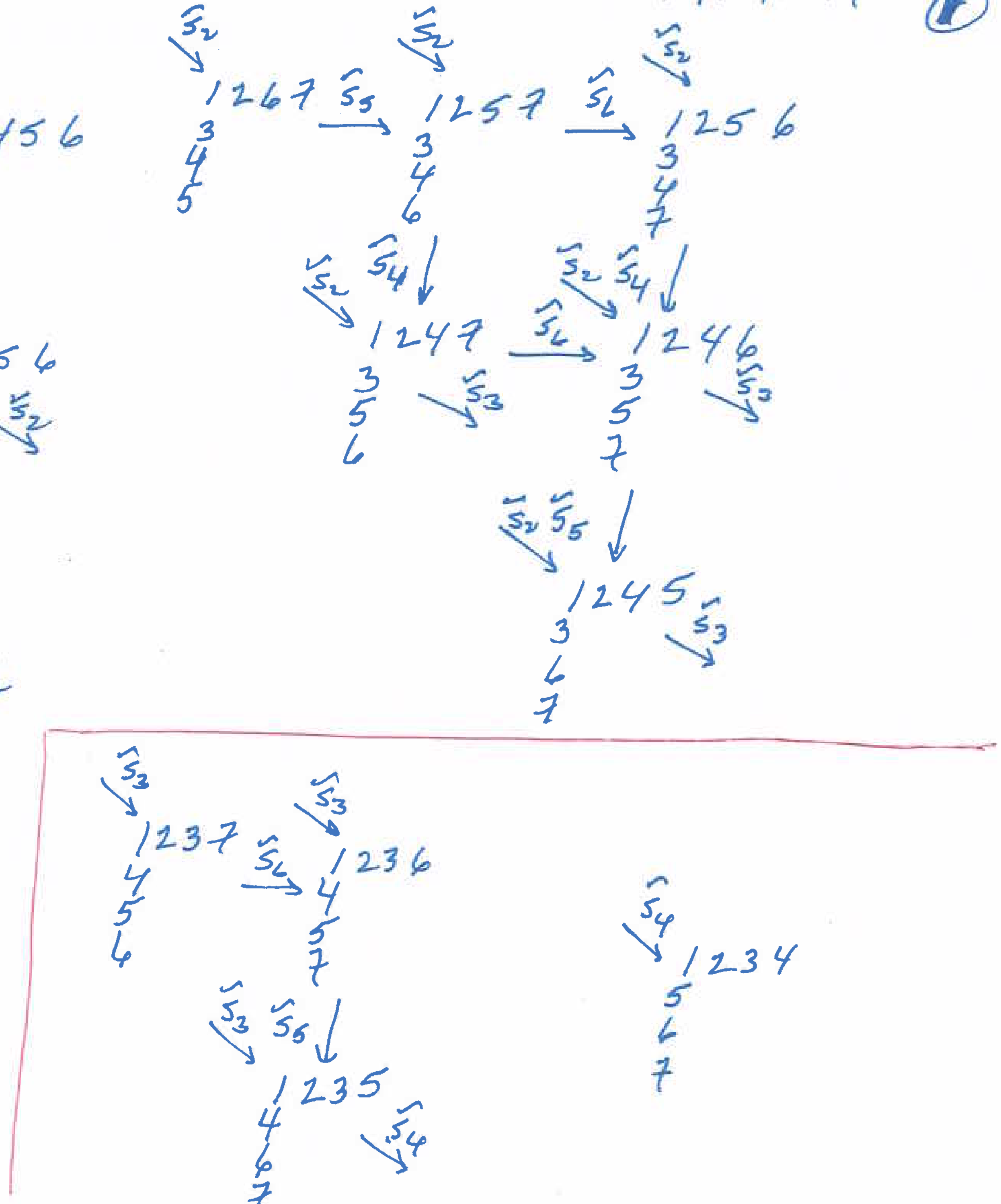
The crystal $B_7^{\boxminus}$

12/12/2017





$$\mathrm{Card}(B_7^{\boxminus}) = \frac{\cancel{7}\cdot\cancel{6}\cdot 5\cdot 4\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{7}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}\cdot\cancel{3}\cdot\cancel{2}\cdot 1} = 20$$

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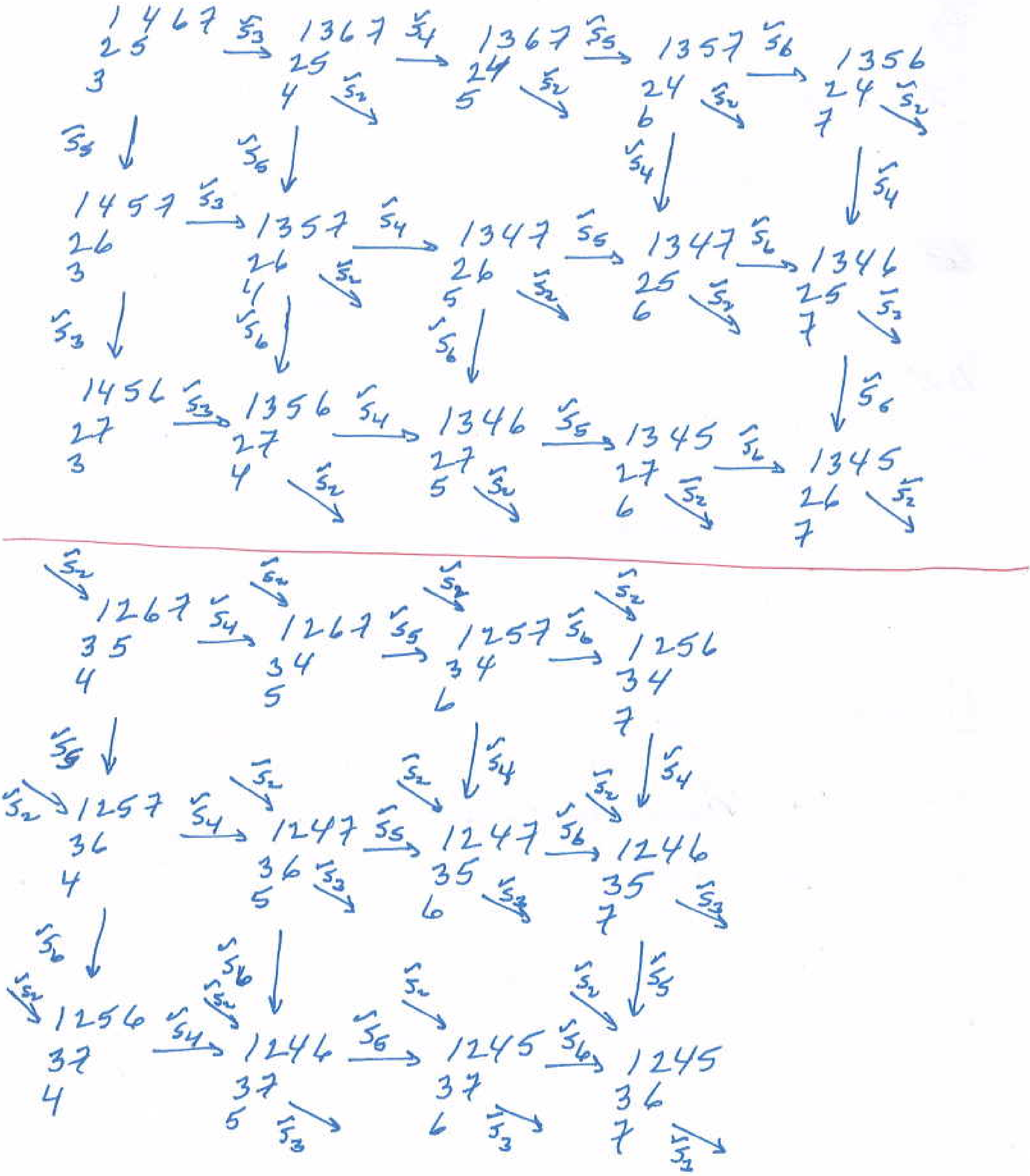
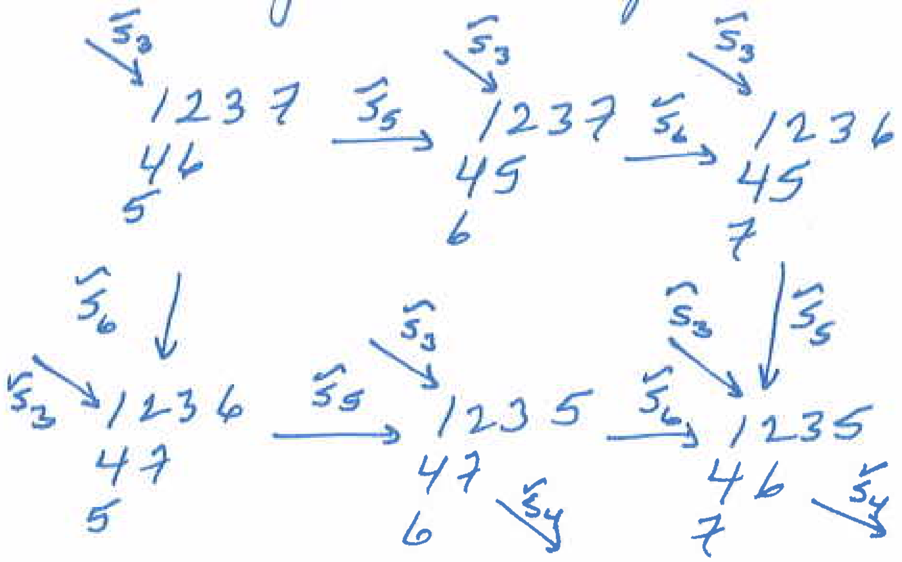
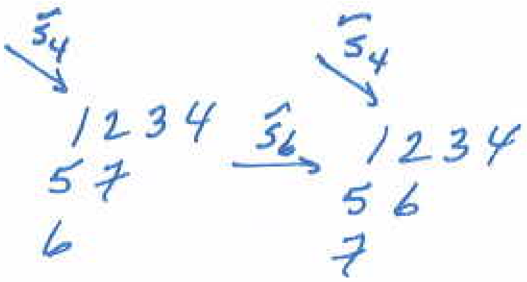
The Hexagon $B_7^{\boxplus}$



The hexagon B_7^{P} page 2.





$$\mathrm{Card}(B_7^{P}) = \frac{7\cdot\cancel{6}\cdot 5\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{1\cdot\cancel{3}\cdot 1\cdot\cancel{6}\cdot\cancel{4}\cdot\cancel{2}\cdot\cancel{1}} = 35$$

The crystal $B_7^{\boxplus}$

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$$
\begin{tikzcd}
\begin{smallmatrix}1&3&5&7\\2&4&6\end{smallmatrix} \arrow[rr, "\check{s}_6"] \arrow[dd, "\check{s}_4"'] \arrow[dr, "\check{s}_2"] & & \begin{smallmatrix}1&3&5&6\\2&4&7\end{smallmatrix} \arrow[dd, "\check{s}_4"'] \arrow[dr, "\check{s}_2"] & \\
& {} & & {} \\
\begin{smallmatrix}1&3&4&7\\2&5&6\end{smallmatrix} \arrow[rr, "\check{s}_6"] \arrow[dr, "\check{s}_2"] & & \begin{smallmatrix}1&3&4&6\\2&5&7\end{smallmatrix} \arrow[dd, "\check{s}_5"'] \arrow[dr, "\check{s}_2"] & \\
& {} & & {} \\
& & \begin{smallmatrix}1&3&4&5\\2&6&7\end{smallmatrix} \arrow[dr, "\check{s}_2"] & \\
& & & {}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\begin{smallmatrix}1&2&5&7\\3&4&6\end{smallmatrix} \arrow[rr, "\check{s}_6"] \arrow[dd, "\check{s}_4"'] & & \begin{smallmatrix}1&2&5&6\\3&4&7\end{smallmatrix} \arrow[dd, "\check{s}_4"'] & \\
& & & \\
\begin{smallmatrix}1&2&4&7\\3&5&6\end{smallmatrix} \arrow[rr, "\check{s}_6"] \arrow[dr, "\check{s}_3"] & & \begin{smallmatrix}1&2&4&6\\3&5&7\end{smallmatrix} \arrow[dd, "\check{s}_5"'] \arrow[dr, "\check{s}_3"] & \\
& {} & & {} \\
& & \begin{smallmatrix}1&2&4&5\\3&6&7\end{smallmatrix} \arrow[dr, "\check{s}_3"] & \\
& & & {}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\begin{smallmatrix}1&2&3&7\\4&5&6\end{smallmatrix} \arrow[r, "\check{s}_6"] & \begin{smallmatrix}1&2&3&6\\4&5&7\end{smallmatrix} \arrow[d, "\check{s}_5"'] & \\
& \begin{smallmatrix}1&2&3&5\\4&6&7\end{smallmatrix} \arrow[dr, "s_4"] & \\
& & \begin{smallmatrix}1&2&3&4\\5&6&7\end{smallmatrix}
\end{tikzcd}
$$

$$\mathrm{Card}(B_7^{\boxplus}) = \frac{7\cdot\cancel{6}\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot 2\cdot\cancel{1}}{1\cdot\cancel{2}\cdot\cancel{3}\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{1}} = 14$$

The crystal $B_7^{\boxplus}$

$$\begin{tikzcd}[column sep=small]
\begin{smallmatrix}1&4&5&6&7\\2\\3\end{smallmatrix} \arrow[r, "\check{s}_3"] & \begin{smallmatrix}1&3&5&6&7\\2\\4\end{smallmatrix} \arrow[r, "\check{s}_4"] \arrow[d, "\check{s}_2"] & \begin{smallmatrix}1&3&4&6&7\\2\\5\end{smallmatrix} \arrow[r, "\check{s}_5"] \arrow[d, "\check{s}_2"] & \begin{smallmatrix}1&3&4&5&7\\2\\6\end{smallmatrix} \arrow[r, "\check{s}_6"] \arrow[d, "\check{s}_2"] & \begin{smallmatrix}1&3&4&5&6\\2\\7\end{smallmatrix} \arrow[d, "\check{s}_2"] \\
& \begin{smallmatrix}1&2&5&6&7\\3\\4\end{smallmatrix} \arrow[r, "\check{s}_4"] & \begin{smallmatrix}1&2&4&6&7\\3\\5\end{smallmatrix} \arrow[r, "\check{s}_5"] \arrow[d, "\check{s}_3"] & \begin{smallmatrix}1&2&4&5&7\\3\\6\end{smallmatrix} \arrow[r, "\check{s}_6"] \arrow[d, "\check{s}_3"] & \begin{smallmatrix}1&2&4&5&6\\3\\7\end{smallmatrix} \arrow[d, "\check{s}_3"] \\
& & \begin{smallmatrix}1&2&3&6&7\\4\\5\end{smallmatrix} \arrow[r, "\check{s}_5"] & \begin{smallmatrix}1&2&3&5&7\\4\\6\end{smallmatrix} \arrow[r, "\check{s}_6"] \arrow[d, "\check{s}_4"] & \begin{smallmatrix}1&2&3&5&6\\4\\7\end{smallmatrix} \arrow[d, "\check{s}_4"] \\
& & & \begin{smallmatrix}1&2&3&4&7\\5\\6\end{smallmatrix} \arrow[r, "\check{s}_6"] & \begin{smallmatrix}1&2&3&4&6\\5\\7\end{smallmatrix} \arrow[d, "\check{s}_5"] \\
& & & & \begin{smallmatrix}1&2&3&4&5\\6\\7\end{smallmatrix}
\end{tikzcd}$$

$$\mathrm{Card}(B_7^{\boxplus}) = \frac{\cancel{7}\cdot 6\cdot 5\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}}{1\cdot 2\cdot \cancel{7}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}} = 15$$

The crystal $B_7^{\square\square}$

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$$
\begin{tikzcd}
\begin{array}{l}13567\\24\end{array} \arrow[r, "\tilde{s}_4"] \arrow[d, "\tilde{s}_2"'] & \begin{array}{l}13467\\25\end{array} \arrow[r, "\tilde{s}_5"] \arrow[d, "\tilde{s}_2"'] & \begin{array}{l}13457\\26\end{array} \arrow[r, "\tilde{s}_6"] \arrow[d, "\tilde{s}_2"'] & \begin{array}{l}13456\\27\end{array} \arrow[d, "\tilde{s}_2"'] \\
\begin{array}{l}12567\\34\end{array} \arrow[r, "\tilde{s}_4"] & \begin{array}{l}12467\\35\end{array} \arrow[r, "\tilde{s}_5"] \arrow[d, "\tilde{s}_3"'] & \begin{array}{l}12457\\36\end{array} \arrow[r, "\tilde{s}_6"] \arrow[d, "\tilde{s}_3"'] & \begin{array}{l}12456\\37\end{array} \arrow[d, "\tilde{s}_3"'] \\
 & \begin{array}{l}12367\\45\end{array} \arrow[r, "\tilde{s}_5"] & \begin{array}{l}12357\\46\end{array} \arrow[r, "\tilde{s}_6"] \arrow[d, "\tilde{s}_4"'] & \begin{array}{l}12356\\47\end{array} \arrow[d, "\tilde{s}_4"'] \\
 & & \begin{array}{l}12347\\56\end{array} \arrow[r, "\tilde{s}_6"] & \begin{array}{l}12346\\57\end{array} \arrow[d, "\tilde{s}_5"'] \\
 & & & \begin{array}{l}12345\\67\end{array}
\end{tikzcd}
$$

$$\mathrm{Card}(B_7^{\square\square}) = \frac{7\cdot\cancel{6}\cdot\cancel{5}\cdot 4\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{1\cdot 2\cdot\cancel{6}\cdot\cancel{5}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}} = 14$$

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The crystals $B_7^{\natural}$ and B_7^{ϕ}

$$\underset{2}{}134567 \xrightarrow{\tilde{s}_2} \underset{3}{}124567 \xrightarrow{\tilde{s}_3} \underset{4}{}123567 \xrightarrow{\tilde{s}_4} \underset{5}{}123467 \xrightarrow{\tilde{s}_5} \underset{6}{}123457 \xrightarrow{\tilde{s}_6} \underset{7}{}123456$$

$$\mathrm{Card}(B_7^{\natural}) = \frac{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{7\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = 6.$$

1234567 with $\mathrm{Card}(B_7^{\phi}) = \dfrac{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = 1.$

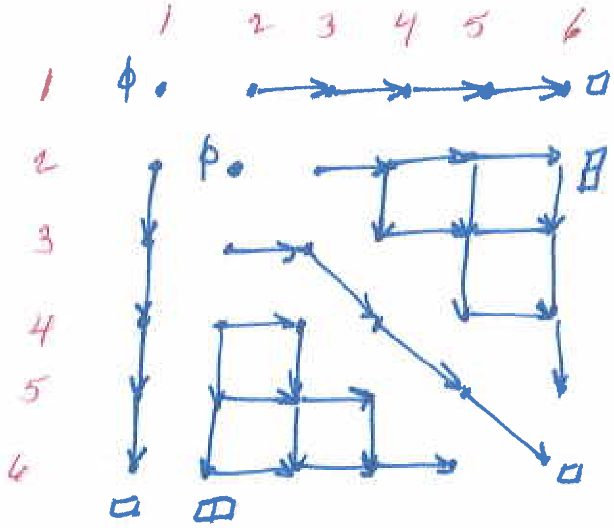
Products $B^{\square} \otimes B^{\square}$

$$B_2^{\square} \otimes B_2^{\square} = B_2^{\phi}$$

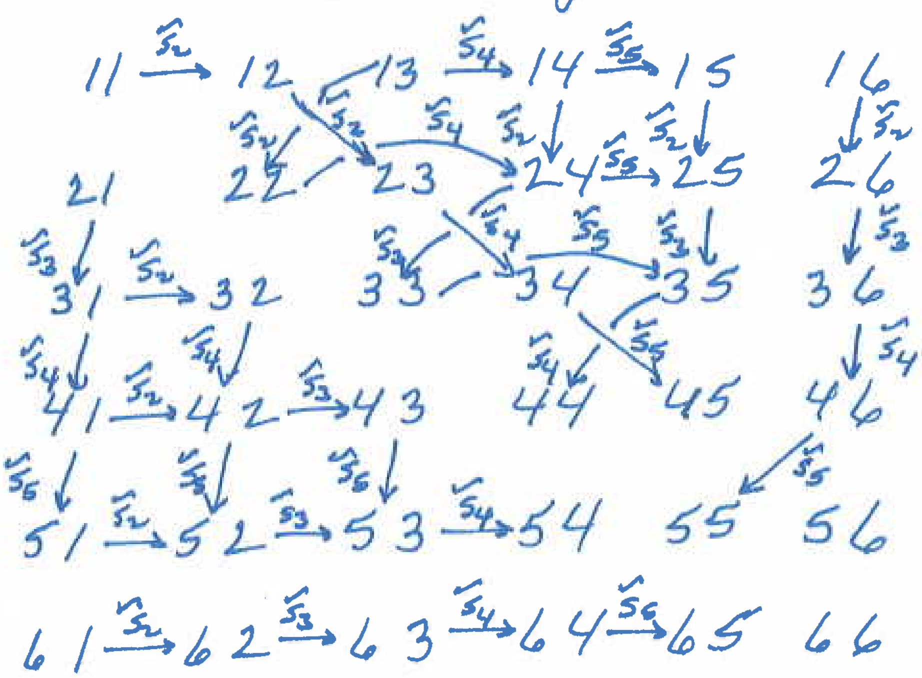
$$B_3^{\square} \otimes B_3^{\square} = B_3^{\phi} + B_3^{\square} + B_3^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$$

$$B_4^{\square} \otimes B_4^{\square} = B_4^{\phi} + B_4^{\square} + B_4^{\square\square} + B_4^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$$

and then it is stable



Halverson-Lewandowski gives



Products $B^{\square} \otimes B^{\square} \otimes B^{\square}$

$$B_2^{\square} \otimes B_2^{\square} \otimes B_2^{\square} = B_2^{\square}$$

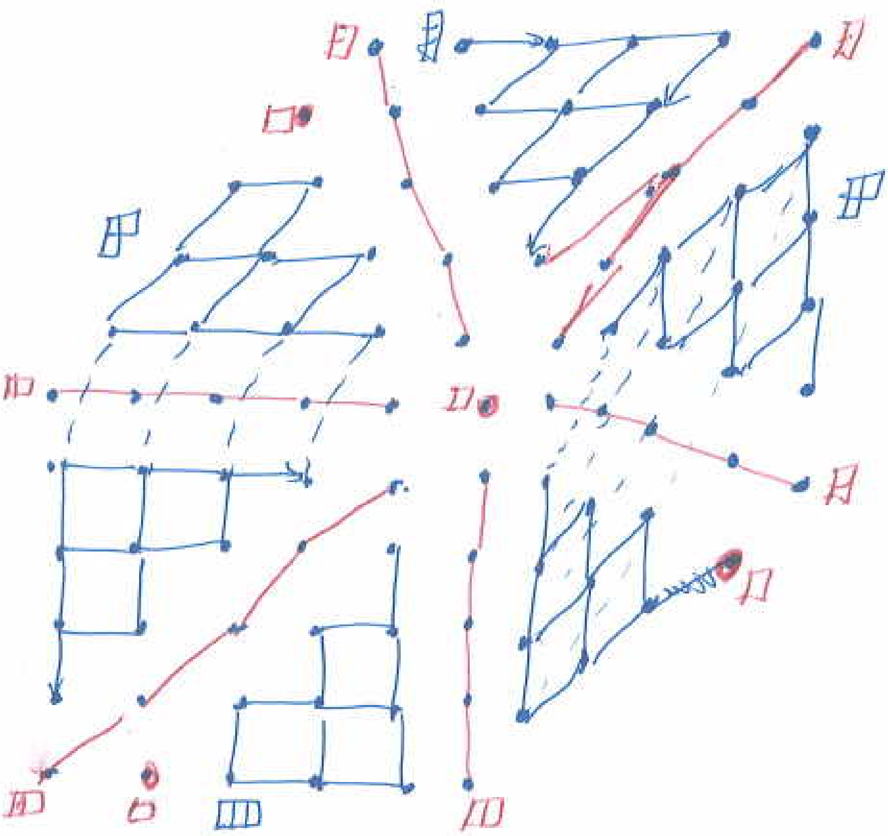
$$B_3^{\square} \otimes B_3^{\square} \otimes B_3^{\square} = B_3^{\phi} + 3B_3^{\square} + B_3^{\boxminus}$$

$$B_4^{\square} \otimes B_4^{\square} \otimes B_4^{\square} = B_4^{\phi} + 4B_4^{\square} + 2B_4^{\square\square\square} + 3B_4^{\boxminus} + B_4^{\boxminus}$$

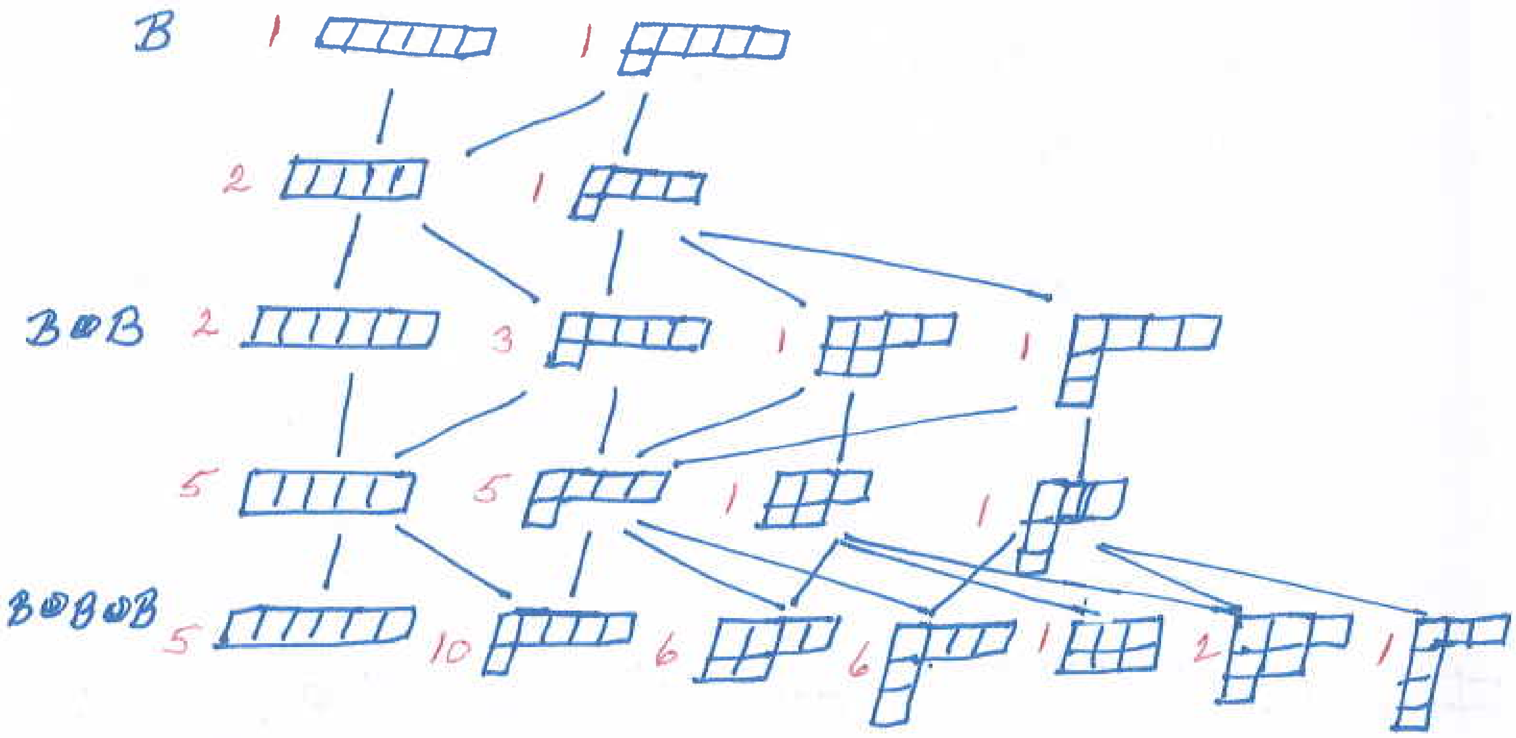
$$B_5^{\square} \otimes B_5^{\square} \otimes B_5^{\square} = B_5^{\phi} + 4B_5^{\square} + 3B_5^{\square\square} + 3B_5^{\boxminus} + 2B_5^{\boxplus} + B_5^{\boxminus}$$

$$B_6^{\square} \otimes B_6^{\square} \otimes B_6^{\square} = B_6^{\phi} + 4B_6^{\square} + 3B_6^{\square\square} + 3B_6^{\boxminus} + B_6^{\square\square\square} + 2B_6^{\boxplus} + B_6^{\boxminus}$$

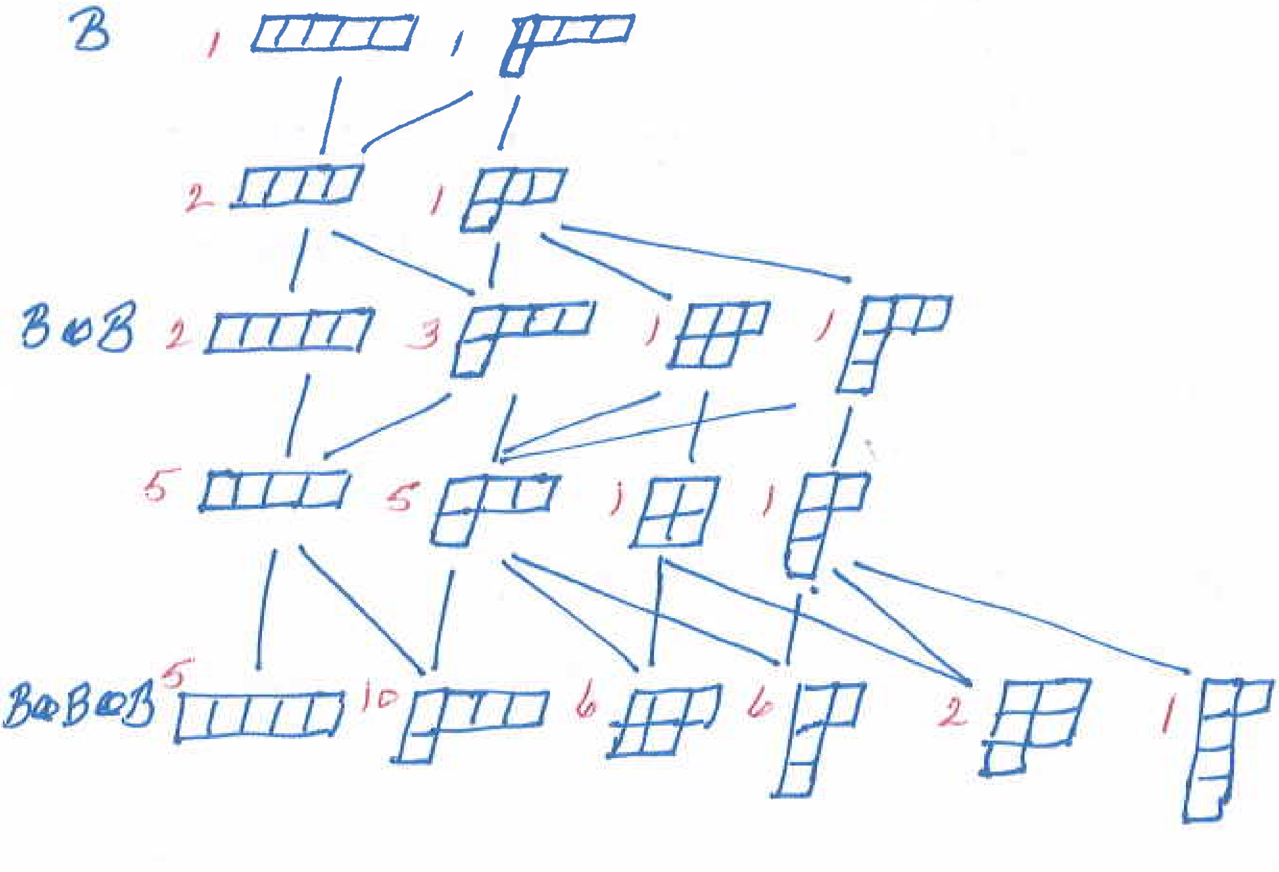
and then it is stable.



Bratelli diagram n=6



Bratelli diagram n=5



Bratelli diagram n=4

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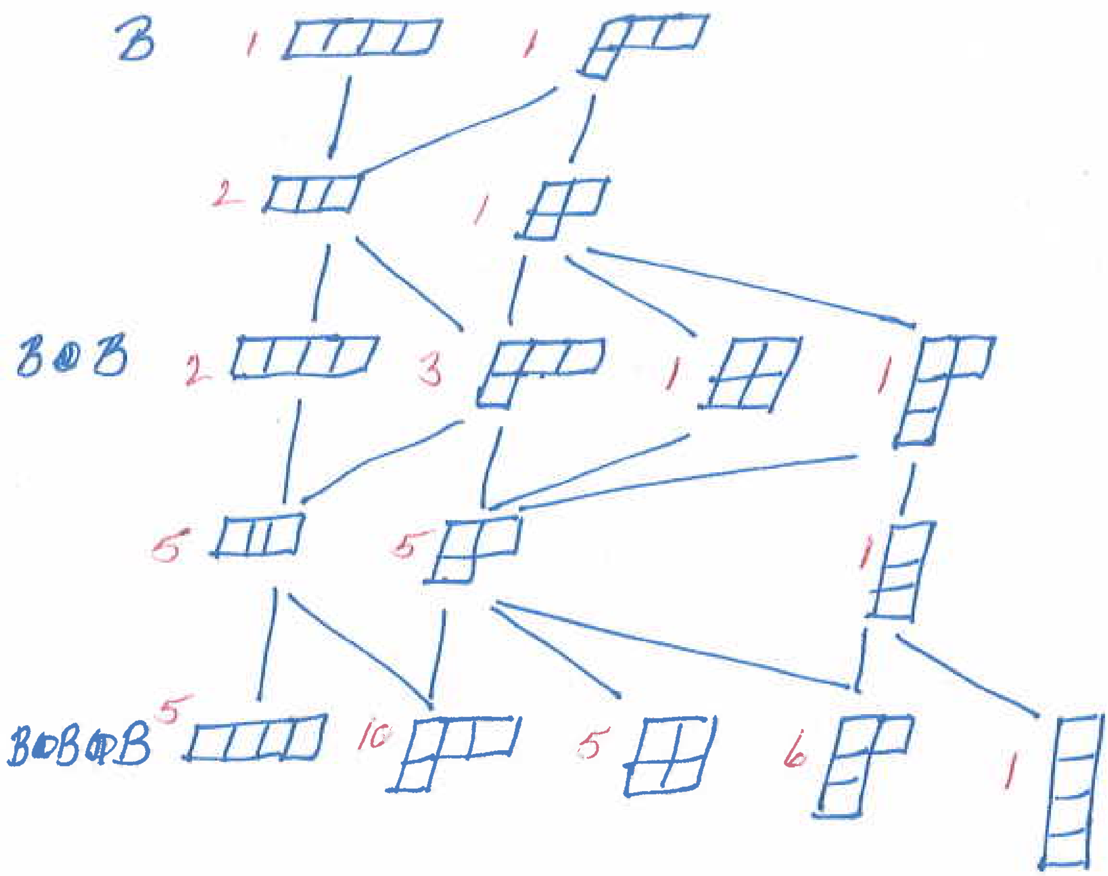
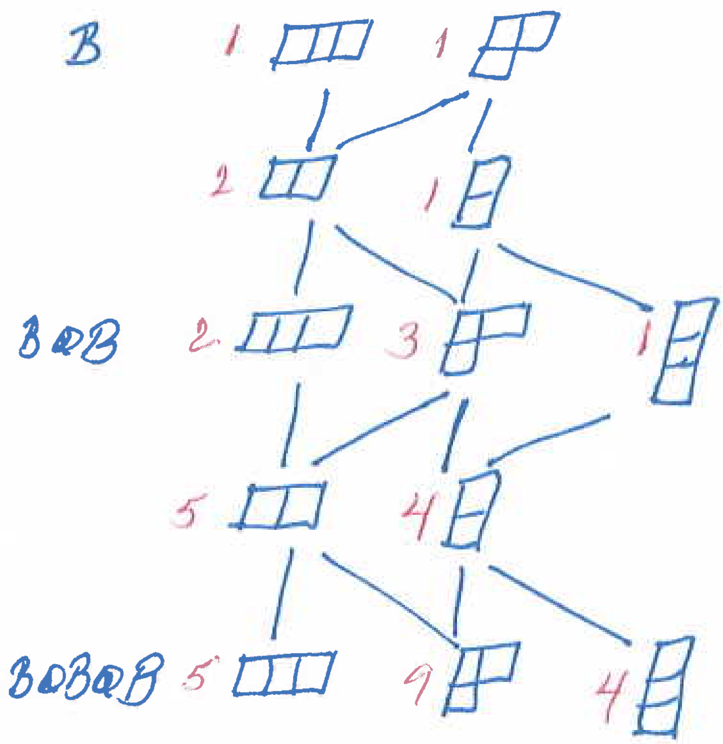
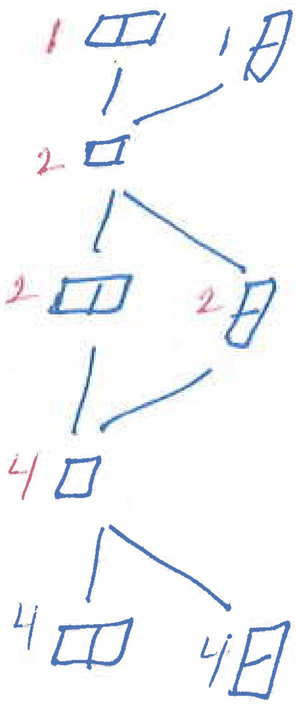


Bratelli diagram n=3



Bratelli diagram n=2



Definitions and remarks

Highest weight: The highest weight of B_n^λ is the column reading tableau

Weights The weight of a standard tableau is $(c(T(1)), \ldots, c(T(n)))$

ξ_i operators $\xi_i \cdot T$ is T with box i and box $i+1$ switched

$$(\xi_i T)(i) = T(i+1)$$
$$(\xi_i T)(i+1) = T(i)$$
$$(\xi_i T)(j) = T(j), \text{ if } j \neq i, i+1.$$

The sign representation: If ~~$\$$~~ is the sign representation then

$$(\lambda \otimes s) \otimes (\mu \otimes s) = (\lambda \otimes \mu) \otimes (s \otimes s) = \lambda \otimes \mu$$
$$(\lambda \otimes s) \otimes \mu = (\lambda \otimes \mu) \otimes s.$$

Goal and bad news

Define B_n^λ and $B_n^\mu \otimes B_n^\nu \xrightarrow{\pi} \bigoplus (B_n^\lambda)^{\oplus \gamma_{\mu\nu}^\lambda}$

with

(a) $\mathrm{Card}(B_n^\lambda) = \dim(S_n^\lambda)$

(b) $\pi(a \otimes \pi(b \otimes c)) = \pi(\pi(a \otimes b) \otimes c)$.

Example $n=3$: Let

$B_3^{(3)} = \{A\}$ $\quad B_3^{(2,1)} = \{B, C\}$ $\quad B_3^{(1,1,1)} = \{D\}$.

We need

$\otimes$	A	B	C	D
A	A	B	C	D
B	B	x	y	r
C	C	z	w	s
D	D	t	u	A

with (x,y,z,w) a permutation of (A,B,C,D)

(r,s) a permutation of (B,C)

(t,u) a permutation of (B,C)

Checks

(1) If $BD = r = C$ and $DB = t = C$ then

$(BD)B = CB$ and $B(DB) = BC$,

but $CB \neq BC$.

Good and bad news continued

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(2) If $BD = C$ and $DB = B$ then
$(BD)B = CB$ and $B(DB) = BB$,
but $CB \neq BB$.

(3) If $BD = B$ and $DB = C$ then
$(BD)B = BB$ and $B(DB) = BC$,
but $BB \neq BC$

Hence $BD = B$ and $DB = B$.
So $CD = C$ and $DC = C$.

(a) If $BB = A$ then $(BB)D = AD = D$
and $B(BD) = BB = A$,
but $D \neq A$.

(b) If $BB = D$ then $(BB)D = DD = A$
and $B(BD) = BB = D$, but $A \neq D$.

(c) If $CC = A$ then $(CC)D = AD = D$
and $C(CD) = CC = A$, but $A \neq D$.

(d) If $CC = D$ then $(CC)D = DD = A$
and $C(CD) = CC = D$, but $A \neq D$

So $BC = A$ or $BC = D$.

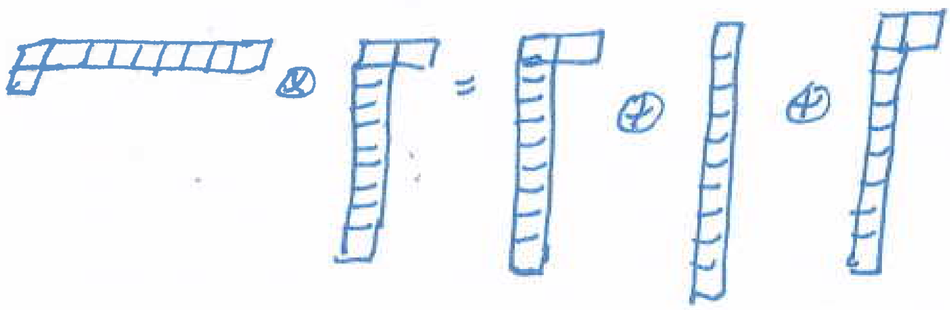
Goal and bad news

(e) If $BC = A$ then $(BC)D = AD = D$
and $B(CD) = BC = A$, but $D \neq A$.

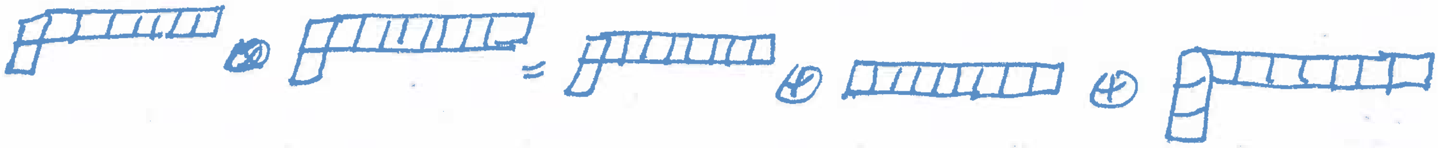
(f) If $BC = D$ then $(BC)D = DD = A$
and $B(CD) = BC = D$, but $A \neq D$.

So the S_3-crystal is impossible //.

For future use



and



The crystal for S_2 does exist

$B_2^{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = \begin{smallmatrix}1\\2\end{smallmatrix}$ and $B_2^{\square\square} = 12$ and

$(B_2)^{\otimes k} = \{ i_1 i_2 \cdots i_\ell \mid i_j \in \{1\,2\} \}$ and

$$i_1 i_2 \cdots i_\ell = \begin{cases} 12, & \text{if } \#2\text{s in } i_1 \cdots i_\ell \text{ is even} \\ \begin{smallmatrix}1\\2\end{smallmatrix}, & \text{if } \#2\text{s in } i_1 \cdots i_\ell \text{ is odd.} \end{cases}$$