Are there Sn crystals?

Representation theory of symmetric groups and related algebras
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The Kronecker problem

S_n^λ the simple $\mathbb{C}S_n$-module indexed by $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ where $\lambda_i \in \mathbb{Z}_{\geq 0}$ and $\lambda_1 + \lambda_2 + \cdots = n$.

Compute $g_{\mu\nu}^\lambda$, where

$$S_n^\mu \otimes S_n^\nu = \bigoplus_\lambda \left(S_n^\lambda\right)^{\oplus g_{\mu\nu}^\lambda}$$

Littlewood-Richardson coefficients

$L_n(\lambda)$ the finite dim. simple $GL_n(\mathbb{C})$ module indexed by $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$ with $\lambda_i \in \mathbb{Z}_{\geq 0}$ and $\lambda_{n+1} = 0$.

Compute $c_{\mu\nu}^{\lambda}$ where

$$L_n(\mu) \otimes L_n(\nu) = \bigoplus_{\lambda} L_n(\lambda)^{\oplus c_{\mu\nu}^{\lambda}}$$

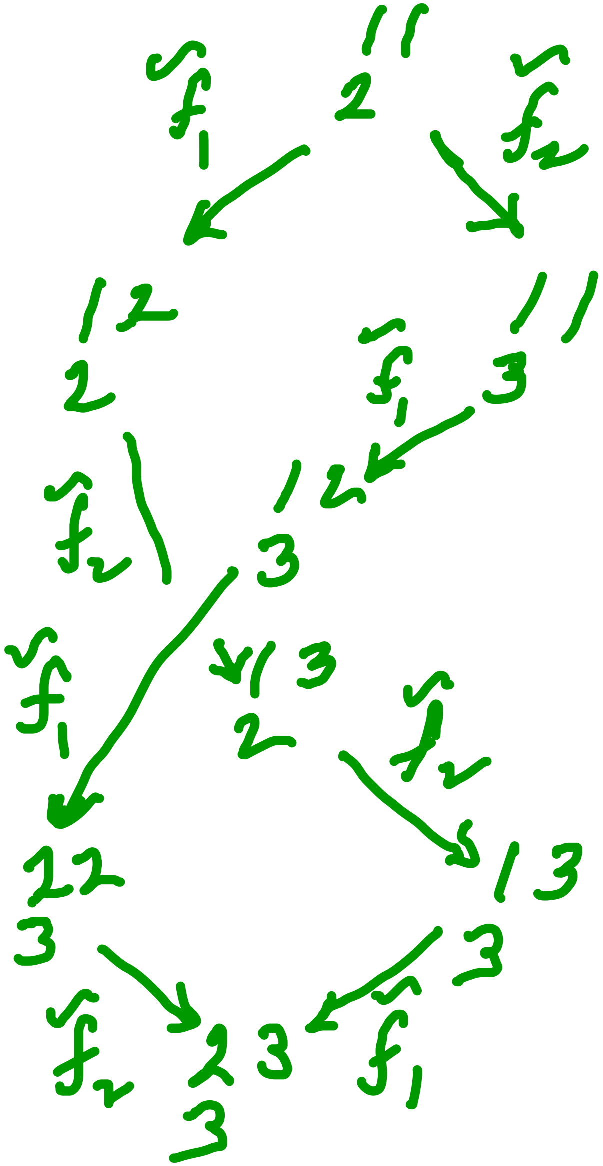
Also the same for reductive alg. G.

The Littlewood-Richardson rule

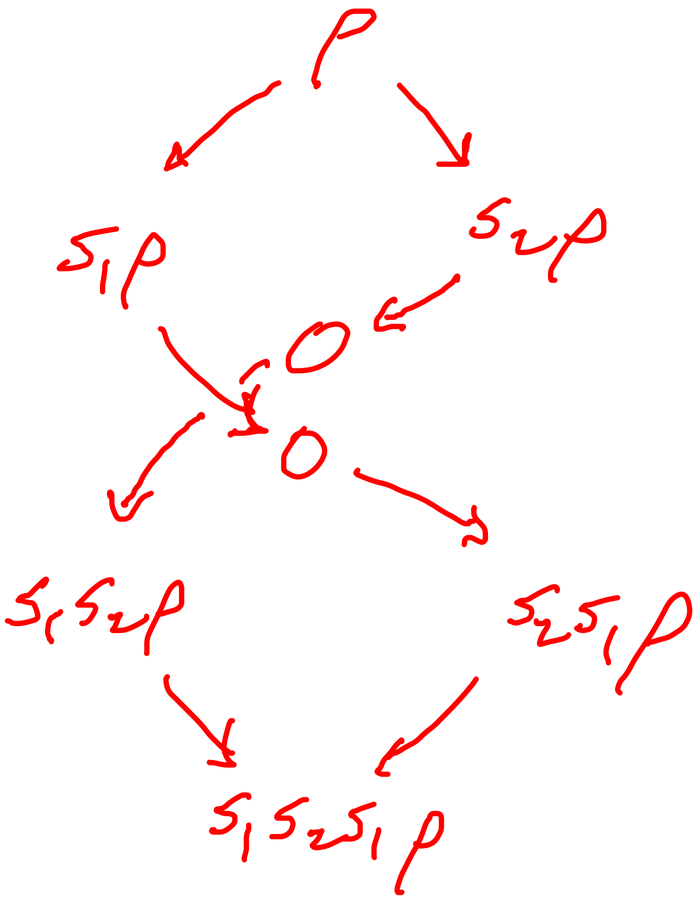
$B_n(\lambda)$ the crystal graph of $L_n(\lambda)$.

$c_{\mu\nu}^{\lambda}$ = # of connected components of highest weight λ in the labeled graph $B_n(\mu) \otimes B_n(\nu)$

Example $B_3(\rho)$ for $GL_3(\mathbb{C})$



Weights



Example $B_5^{\boxplus}$ for $\mathbb{C}S_5$

$$
\begin{tikzcd}
& \begin{array}{cc} 1 & 4 \\ 2 & 5 \\ 3 & \end{array} \arrow[d, "\tilde{s}_3"'] & \\
& \begin{array}{cc} 1 & 3 \\ 2 & 5 \\ 4 & \end{array} \arrow[dl, "\tilde{s}_2"'] \arrow[dr, "\tilde{s}_4"] & \\
\begin{array}{cc} 1 & 2 \\ 3 & 5 \\ 4 & \end{array} \arrow[dr, "\tilde{s}_4"'] & & \begin{array}{cc} 1 & 3 \\ 2 & 4 \\ 5 & \end{array} \arrow[dl, "\tilde{s}_2"] \\
& \begin{array}{cc} 1 & 2 \\ 3 & 4 \\ 5 & \end{array} &
\end{tikzcd}
$$

Weights

$$
\begin{tikzcd}
& (0,-1,-2,1,0) \arrow[d] & \\
& (0,-1,1,-2,0) \arrow[dl] \arrow[dr] & \\
(0,1,-1,-2,0) \arrow[dr] & & (0,-1,1,0,-2) \arrow[dl] \\
& (0,1,-1,0,-2) &
\end{tikzcd}
$$

The <u>highest</u> <u>weight</u> is the column reading tableau

Tensor categories - Graph categorification

$B_n = \{$certain labeled graphs$\}$

$\oplus$ is disjoint union $\sqcup$

simple objects B_n^λ for $\lambda \vdash n$
which are the connected graphs in B_n

$$\mathrm{Card}(B_n^\lambda) = \dim(S_n^\lambda)$$

monoidal structure

$$\pi : B_n^\mu \otimes B_n^\nu \longrightarrow \coprod_\lambda (B_n^\lambda)^{\sqcup \gamma_{\mu\nu}^\lambda}$$

with

$$\pi(\pi(a\otimes b)\otimes c) = \pi(a\otimes \pi(b\otimes c))$$

Grothendieck ring

$$K(B_n) \xrightarrow{\sim} K(\mathbb{C}S_n\text{-mod})$$

$$[B_n^\lambda] \longmapsto [S_n^\lambda]$$

<u>monoidal structure</u>

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<u>Theorem</u>

monoidal structure

$$\pi : B_n^\mu \otimes B_n^\nu \longrightarrow \bigsqcup_\lambda (B_n^\lambda)^{\sqcup \gamma_{\mu\nu}^\lambda}$$

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Grothendieck ring

$$K(B_n) \xrightarrow{\sim} K(\mathbb{C}S_n\text{-mod})$$

$$[B_n^\lambda] \longmapsto [S_n^\lambda]$$

Theorem If $n \geq 3$ then B_n does not exist.

Example $n=3$

$\otimes$	$S_3^{(3)}$	$S_3^{(2,1)}$	$S_3^{(1,1,1)}$
$S_3^{(3)}$	$S_3^{(3)}$	$S_3^{(2,1)}$	$S_3^{(1,1,1)}$
$S_3^{(2,1)}$	$S_3^{(2,1)}$	$S_3^{(3)} \oplus S_3^{(2,1)} \oplus S_3^{(1,1,1)}$ $S_3^{(2,1)}$	
$S_3^{(1,1,1)}$	$S_3^{(1,1,1)}$	$S_3^{(2,1)}$	$S_3^{(3)}$

needs to be expanded

Let

$A = 1\,2\,3 \qquad B = \begin{array}{ll} 1 & 3 \\ 2 & \end{array} \qquad C = \begin{array}{ll} 1 & 2 \\ 3 & \end{array} \qquad D = \begin{array}{l} 1 \\ 2 \\ 3 \end{array}$

$\otimes$	A	B	C	D
A	A	B	C	D
B	B			
C	C			
D	D			A

permutation of (A, B, C, D)

permutation of (B, C)

with $(xy)z = x(yz)$.

Let

$A = 1\,2\,3 \qquad B = \begin{array}{ll}1 & 3\\ 2 & \end{array} \qquad C = \begin{array}{ll}1 & 2\\ 3 & \end{array} \qquad D = \begin{array}{l}1\\2\\3\end{array}$

$\otimes$	A	B	C	D
A	A	B	C	D
B	B			
C	C			
D	D			A

permutation of (A, B, C, D)

permutation of (B, C)

with $(xy)z = x(yz)$.

IMPOSSIBLE.

<u>Stability</u> (Murnaghan, see Brion 1993)

For $\lambda = (\lambda_1, \lambda_2, \ldots)$ let $\bar{\lambda} = (\lambda_2, \lambda_3, \ldots)$

For $n > 2(\mu_2 + \nu_2)$

$\gamma^{\lambda}_{\mu\nu}$ depends only on $\bar{\lambda}, \bar{\mu}, \bar{\nu}$.

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So look at B_n^{λ} for n large.

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For $n > 2(\mu_2 + \nu_2)$

$g_{\bar{\mu}\bar{\nu}}^{\bar{\lambda}} = \gamma_{\mu\nu}^{\lambda}$ depends only on $\bar{\lambda}, \bar{\mu}, \bar{\nu}$.

So look at B_n^{λ} for n large.

Examples

$$B_n = 1\,2\,3\,4 \cdots n$$

$$B_n = \begin{array}{l} 1\,3\,4 \cdots \\ 2 \end{array} \xrightarrow{s_2} \begin{array}{l} 1\,2\,4\,5 \cdots \\ 3 \end{array} \xrightarrow{s_3} \begin{array}{l} 1\,2\,3\,5 \cdots \\ 4 \end{array} \xrightarrow{s_4} \cdots$$

Examples

$$B_n^{\square\square\square\square\square\square\square\square} = 1\,2\,3\,4\cdots n$$

$$B_n^{\begin{smallmatrix}\square&\square&\square&\square&\cdots\\ \square\end{smallmatrix}} = \begin{smallmatrix}1&3&4&\cdots\\ 2\end{smallmatrix} \xrightarrow{s_2} \begin{smallmatrix}1&2&4&5&\cdots\\ 3\end{smallmatrix} \xrightarrow{s_3} \begin{smallmatrix}1&2&3&5&\cdots\\ 4\end{smallmatrix} \xrightarrow{s_4} \cdots$$

$$B_n^{\infty} = \bullet$$

$$B_n^{\square} = \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \cdots$$

$$B_n = \begin{matrix} 1\,3\,5\,6\,7\cdots \\ 2\,4 \end{matrix} \xrightarrow{\tilde{s}_4} \begin{matrix} 1\,3\,4\,6\,7\cdots \\ 2\,5 \end{matrix} \xrightarrow{\tilde{s}_6} \begin{matrix} 1\,3\,5\,7\cdots \\ 2\,6 \end{matrix} \xrightarrow{\tilde{s}_6} \cdots$$

$$
B_n = \begin{tikzcd}
\begin{smallmatrix} 1 & 3 & 5 & 6 & 7 & \cdots \\ 2 & 4 \end{smallmatrix} \arrow[r, "\tilde{s}_4"] \arrow[d, "\tilde{s}_2"'] & \begin{smallmatrix} 1 & 3 & 4 & 6 & 7 & \cdots \\ 2 & 5 \end{smallmatrix} \arrow[r, "\tilde{s}_5"] \arrow[d, "\tilde{s}_2"'] & \begin{smallmatrix} 1 & 3 & 5 & 7 & \cdots \\ 2 & 6 \end{smallmatrix} \arrow[r, "\tilde{s}_6"] \arrow[d, "\tilde{s}_2"'] & \cdots \\
\begin{smallmatrix} 1 & 2 & 5 & 6 & 7 & \cdots \\ 3 & 4 \end{smallmatrix} \arrow[r, "\tilde{s}_4"] & \begin{smallmatrix} 1 & 2 & 4 & 6 & 7 \\ 3 & 5 \end{smallmatrix} \arrow[r, "\tilde{s}_5"] & \begin{smallmatrix} 1 & 2 & 5 & 7 & \cdots \\ 3 & 6 \end{smallmatrix} \arrow[r, "\tilde{s}_6"] & \cdots
\end{tikzcd}
$$

$$
B_n^{\square\square\square\square\cdots} =
\begin{tikzcd}
\begin{smallmatrix} 1\,3\,5\,6\,7\cdots \\ 2\,4 \end{smallmatrix} \arrow[r, "\breve{s}_4"] \arrow[d, "\breve{s}_2"'] & \begin{smallmatrix} 1\,3\,4\,6\,7\cdots \\ 2\,5 \end{smallmatrix} \arrow[r, "\breve{s}_5"] \arrow[d, "\breve{s}_2"'] & \begin{smallmatrix} 1\,3\,4\,5\,7\cdots \\ 2\,6 \end{smallmatrix} \arrow[r, "\breve{s}_6"] \arrow[d, "\breve{s}_2"'] & \cdots \\
\begin{smallmatrix} 1\,2\,5\,6\,7\cdots \\ 3\,4 \end{smallmatrix} \arrow[r, "\breve{s}_4"] & \begin{smallmatrix} 1\,2\,4\,6\,7 \\ 3\,5 \end{smallmatrix} \arrow[r, "\breve{s}_5"] \arrow[d, "\breve{s}_3"'] & \begin{smallmatrix} 1\,2\,4\,5\,7\cdots \\ 3\,6 \end{smallmatrix} \arrow[r, "\breve{s}_6"] \arrow[d, "\breve{s}_3"'] & \cdots \\
& \begin{smallmatrix} 1\,2\,3\,6\,7 \\ 4\,5 \end{smallmatrix} \arrow[r, "\breve{s}_5"] & \begin{smallmatrix} 1\,2\,3\,5\,7 \\ 4\,6 \end{smallmatrix} \arrow[r, "\breve{s}_6"] & \cdots
\end{tikzcd}
$$

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B_n = \begin{tikzcd}
\begin{smallmatrix}1&3&5&6&7&\cdots\\2&4\end{smallmatrix} \arrow[r, "\tilde{s}_4"] \arrow[d, "\tilde{s}_2"'] & \begin{smallmatrix}1&3&4&6&7&\cdots\\2&5\end{smallmatrix} \arrow[r, "\tilde{s}_5"] \arrow[d, "\tilde{s}_2"'] & \begin{smallmatrix}1&3&4&5&7&\cdots\\2&6\end{smallmatrix} \arrow[r, "\tilde{s}_6"] \arrow[d, "\tilde{s}_2"'] & \cdots \\
\begin{smallmatrix}1&2&5&6&7&\cdots\\3&4\end{smallmatrix} \arrow[r, "\tilde{s}_4"] & \begin{smallmatrix}1&2&4&6&7\\3&5\end{smallmatrix} \arrow[r, "\tilde{s}_5"] \arrow[d, "\tilde{s}_3"'] & \begin{smallmatrix}1&2&4&5&7&\cdots\\3&6\end{smallmatrix} \arrow[r, "\tilde{s}_6"] \arrow[d, "\tilde{s}_3"'] & \cdots \\
& \begin{smallmatrix}1&2&3&6&7\\4&5\end{smallmatrix} \arrow[r, "\tilde{s}_5"] & \begin{smallmatrix}1&2&3&5&7\\4&6\end{smallmatrix} \arrow[r, "\tilde{s}_6"] & \cdots
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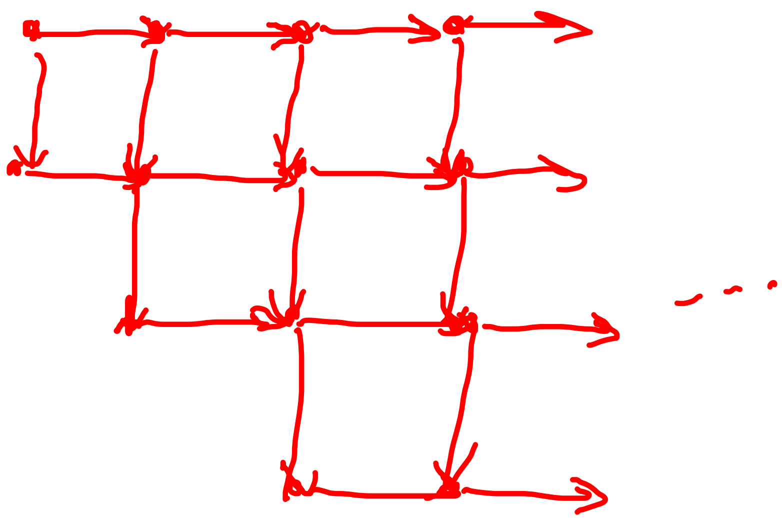
$$

$B_\infty =$



$$B_n = \begin{array}{llll} 1 & 4 & 5 & 6 \cdots \\ 2 \\ 3 \end{array} \xrightarrow{\tilde{s}_3} \begin{array}{llll} 1 & 3 & 5 & 6 \cdots \\ 2 \\ 4 \end{array} \xrightarrow{\tilde{s}_4} \begin{array}{llll} 1 & 3 & 4 & 6 \cdots \\ 2 \\ 5 \end{array} \xrightarrow{\tilde{s}_5} \begin{array}{llll} 1 & 3 & 4 & 5 \cdots \\ 2 \\ 6 \end{array} \xrightarrow{\tilde{s}_6} \cdots$$

$$
\begin{tikzcd}
B_n = \begin{smallmatrix}1&4&5&6&\cdots\\2\\3\end{smallmatrix} \arrow[r, "\tilde{s}_3"] & \begin{smallmatrix}1&3&5&6&\cdots\\2\\4\end{smallmatrix} \arrow[r, "\tilde{s}_4"] \arrow[d, "\tilde{s}_2"'] & \begin{smallmatrix}1&3&4&6&\cdots\\2\\5\end{smallmatrix} \arrow[r, "\tilde{s}_5"] \arrow[d, "\tilde{s}_2"'] & \begin{smallmatrix}1&3&4&5&\cdots\\2\\6\end{smallmatrix} \arrow[r, "\tilde{s}_6"] \arrow[d, "\tilde{s}_2"'] & \cdots \\
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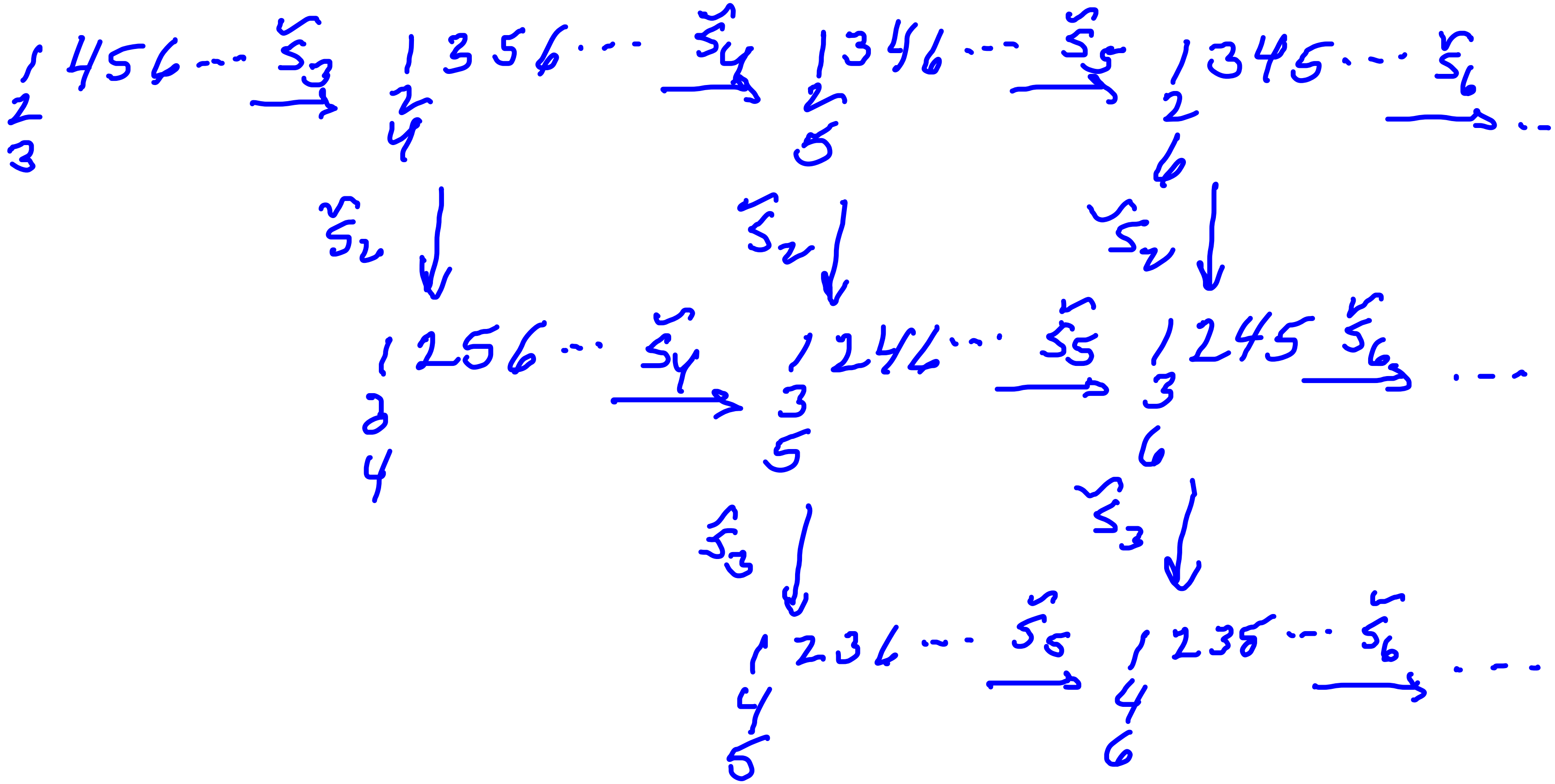
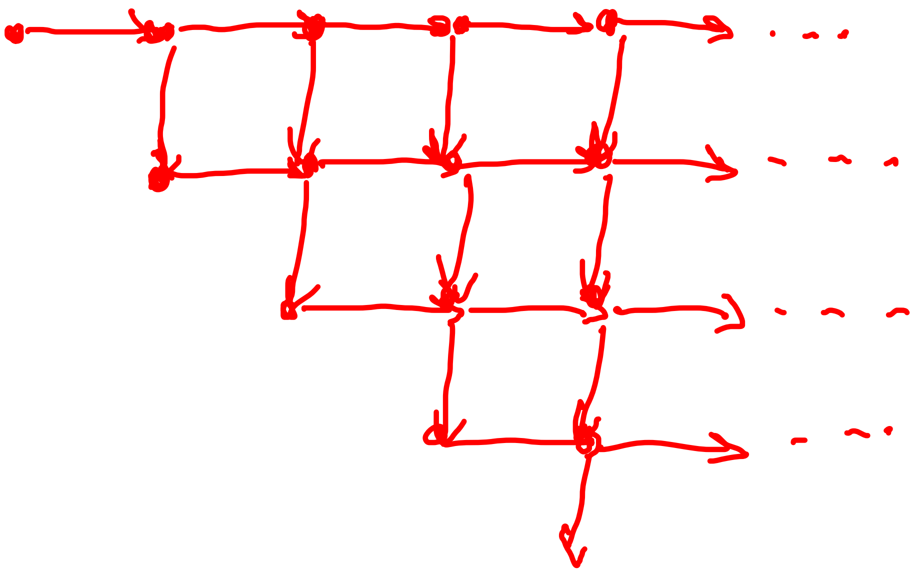
\end{tikzcd}
$$

$B_n^{\lambda} =$



$B_\infty^{\lambda} =$



Define

$B_\infty^{\bar\lambda}$ to be the stable limit of B_n^{λ}

$\dim(B_\infty^{\bar\lambda}) = \#$ of boxes of $\bar\lambda$

$\mathrm{dir}(B_\infty^{\bar\lambda}) = \#$ of removable boxes of $\bar\lambda$.

<u>TODO</u>: Make a graphical tensor category B_∞

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simple objects

$B_\infty^{\bar\lambda}$ for $\bar\lambda = (\lambda_2 \geq \lambda_3 \geq \dots)$ with $\lambda_i \in \mathbb{Z}_{\geq 0}$

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① Use it to compute $\varphi_{\bar\mu \bar\nu}^{\bar\lambda}$

Define

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simple objects

$B_\infty^{\bar\lambda}$ for $\bar\lambda = (\lambda_2 \geq \lambda_3 \geq \cdots)$ with $\lambda_i \in \mathbb{Z}_{\geq 0}$

(1) Use it to compute $\varphi_{\bar\mu \bar\nu}^{\bar\lambda}$

(2) Define $\Gamma_n : B_\infty \longrightarrow B_n$

and use these to compute $\gamma_{\mu\nu}^{\lambda}$

(see Sam-Snowden, ..., Inna Entova-Aizenbud).

Halverson-Lewandowski and Geometry

① As $GL_n(\mathbb{C}) \times S_k$-modules

$$V^{\otimes k} \cong \bigoplus_{\substack{\lambda \vdash k \\ \lambda_{n+1} = 0}} L_n(\lambda) \otimes S_k^\lambda$$

where $V = \mathbb{C}^n$.

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$$\left\{\begin{matrix}\text{words of length } k \\ \text{in } \{1,2,\dots,n\}\end{matrix}\right\} \longleftrightarrow \bigsqcup_{\substack{\lambda \vdash k \\ \lambda_{n+1}=0}} B_n(\lambda) \times B_k^{\lambda}$$

$$i_1 i_2 \cdots i_k \longmapsto (P, Q)$$

Halverson-Lewandowski and Geometry

① As $GL_n(\mathbb{C}) \times S_k$-modules

$$V^{\otimes k} \simeq \bigoplus_{\substack{\lambda \vdash k \\ \lambda_{n+1}=0}} L_n(\lambda) \otimes S_k^\lambda$$

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column strict shape λ

$$i_1 i_2 \cdots i_k \longmapsto (P, Q)$$

standard shape λ

Halverson-Lewandowski and geometry

(2) As $S_n \times P_k(n)$-modules

$$V^{\otimes k} \cong \bigoplus_{\substack{\lambda \vdash n \\ |\bar{\lambda}| \leq k}} S_n^{\lambda} \times P_k(n)^{\lambda}$$

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Halverson-Lewandowski and geometry

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$$i_1 i_2 \cdots i_k \longmapsto (X, Y)$$

X: standard shape λ

Y: remove-add tableau of length $2k$ and shape λ.

Examples: Halverson-Lewandorski with $k=2$

11	12	13	14	15	16
21	22	23	24	25	26
31	32	33	34	35	36
41	42	43	44	45	46
51	52	53	54	55	56
61	62	63	64	65	66

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$$V \otimes V \cong 2S_6^{\emptyset} + 3S_6^{\square} + S_6^{\square\square} + S_6^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$$

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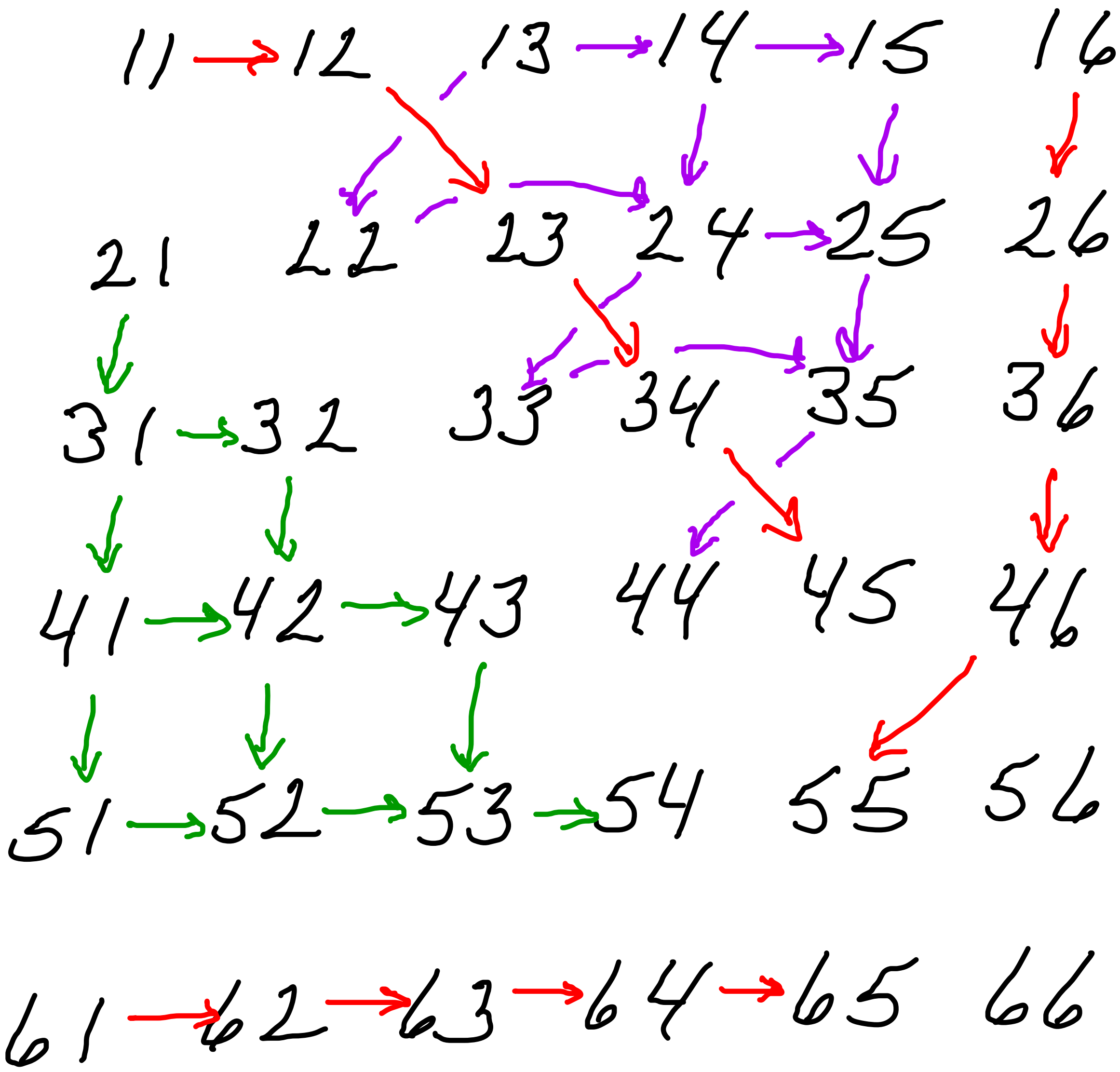
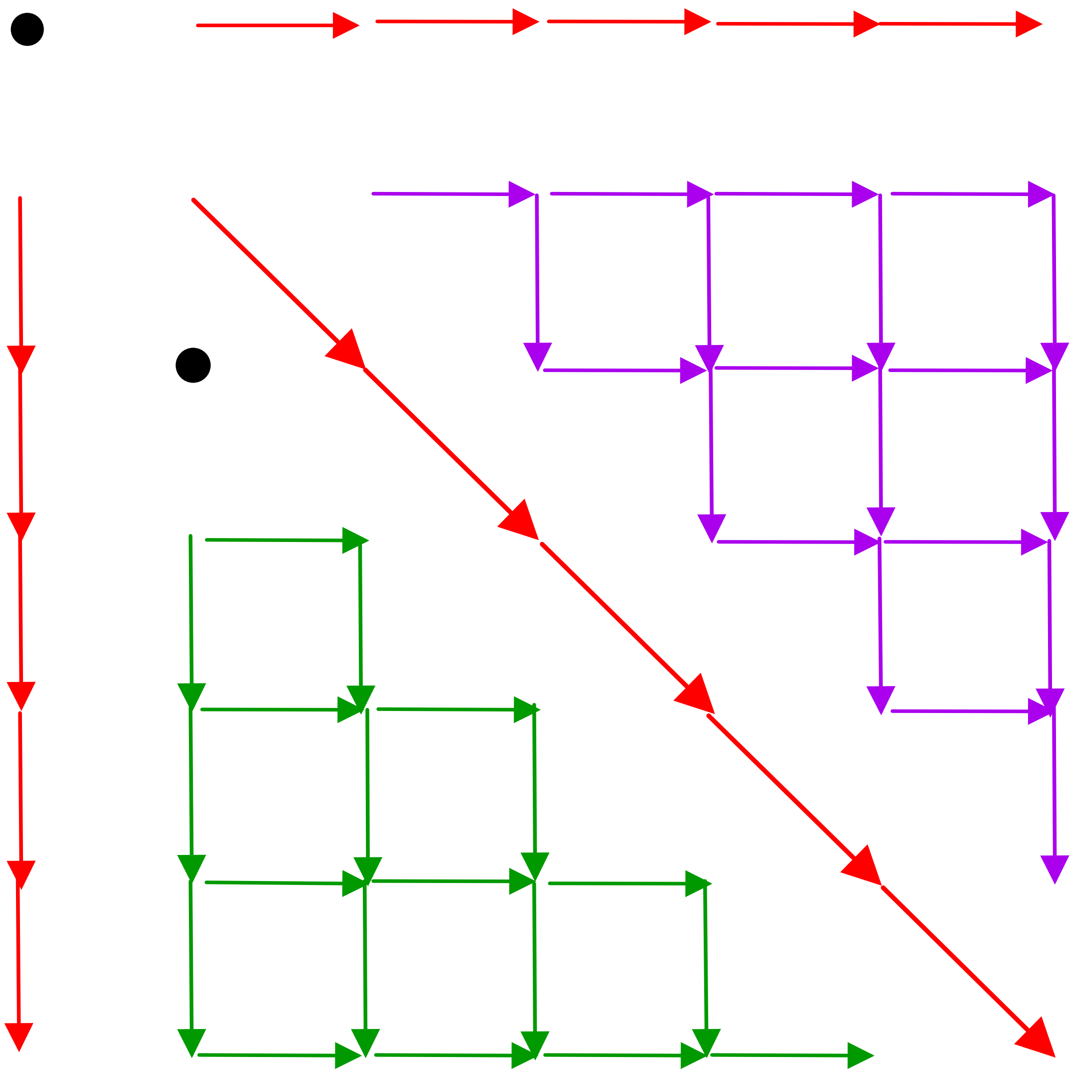


$$V \otimes V = 2S_6^{\emptyset} + 3S_6^{\square} + S_6^{\square\square} + S_6^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$$

My "improved" version



$$V \otimes V \simeq 2 S_6^{\phi} + 3 S_6^{\square} + S_6^{\square\square} + S_6^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$$