Future directions in Representation Theory, Univ. Sydney

Combinatorics of level 0 representations

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Theorem (Kato-Naito-Sagaki) In $K_{I^+ \rtimes \mathbb{C}^\times}(G/I^0)$

arXiv: 1702.02408

$$[\underbrace{\mathcal{L}(\lambda)}_{\text{line bundle}}][\underbrace{\mathcal{O}_{\overline{I^+ w I^0}}}_{\text{semi-infinite Schubert variety}}] = \sum_{p \in \underbrace{\mathcal{B}(\lambda + 0\Lambda_0)_{\geq w}}_{\text{Demazure crystal}}} e^{\mathrm{end}(p)} [\mathcal{O}_{\overline{I^+ \underbrace{\varphi(p)}_{\text{fin. direction of } p} I^0}}]$$

$G = G(\mathbb{C}[\varepsilon, \varepsilon^{-1}])$

I^+ pos. level. Iwahori subgroup

I^0 level 0 Iwahori subgroup

I^- neg. level. Iwahori subgroup

G/I^+ pos. level aff. flag variety

G/I^0 level 0 aff. flag variety

G/I^- neg level. aff. flag variety

$$G = \bigsqcup_{x \in W} I^+ x I^+ = \bigsqcup_{y \in W} I^+ y I^0 = \bigsqcup_{z \in W} I^+ z I^-$$

where

W = affine Weyl group.

Bruhat order

$x \le w$ if $I^+ x I^+ \subseteq \overline{I^+ w I^+}$

$x \le w$ if $I^+ x I^0 \subseteq \overline{I^+ w I^0}$

$x \le w$ if $I^+ x I^- \subseteq \overline{I^+ w I^-}$

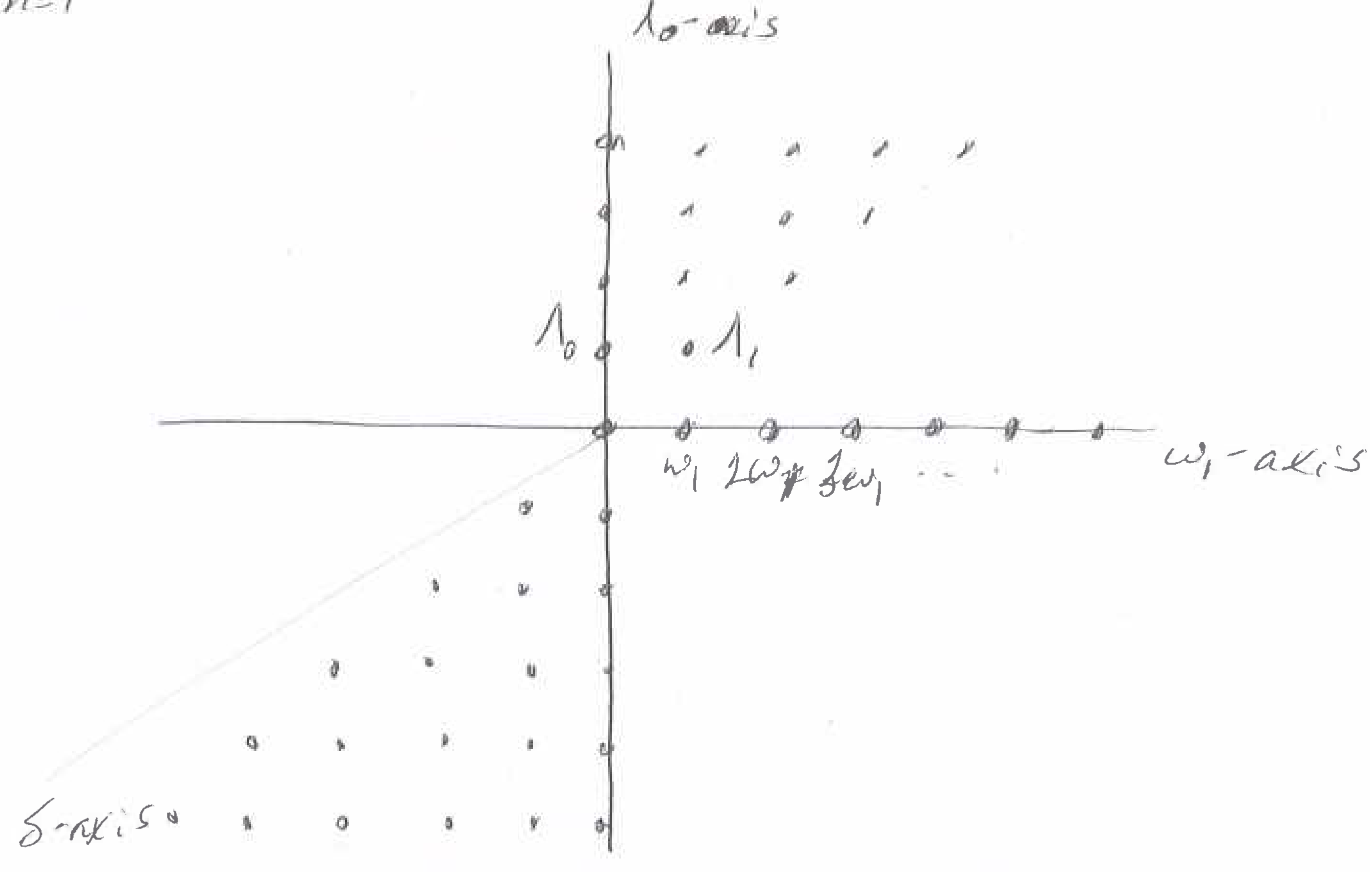
The affine Kac-Moody Lie algebra

$$
\begin{aligned}
\mathfrak{g} &= \mathrm{Lie}(G[\epsilon,\epsilon^{-1}]) \oplus \mathbb{C}K \oplus \mathbb{C}d \\
&\cup| \\
\mathfrak{b} &= \mathrm{Lie}(I^+) \oplus \mathbb{C}K \oplus \mathbb{C}d \\
&\cup| \\
\mathfrak{h} &= \mathrm{Lie}(T(\mathbb{C})) \oplus \mathbb{C}K \oplus \mathbb{C}d .
\end{aligned}
$$

$$\mathfrak{h}^* = \mathrm{span}\{\omega_1, \ldots, \omega_n\} \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta .$$

Integrable $\mathfrak{g}$-modules are indexed by $\Lambda \in (\mathfrak{h}^*)_{\mathrm{int}}$.

$n=1$



Let $\Lambda \in (\mathfrak{h}^*)_{int}$. The extremal weight module

$L(\Lambda)$ is generated by $\{u_{w\Lambda} \mid w \in W\}$ with

$u_{w\Lambda}$ a wt. vector of weight $w\Lambda$.

$e_i u_{w\Lambda} = 0$ and $f_i^{\langle w\Lambda, \alpha_i^\vee\rangle} u_{w\Lambda} = u_{s_i w\Lambda}$, if $\langle w\Lambda, \alpha_i^\vee\rangle \in \mathbb{Z}_{\geq 0}$

$f_i u_{w\Lambda} = 0$ and $e_i^{-\langle w\Lambda, \alpha_i^\vee\rangle} u_{w\Lambda} = u_{s_i w\lambda}$. if $\langle w\Lambda, \alpha_i^\vee\rangle \in \mathbb{Z}_{\leq 0}$

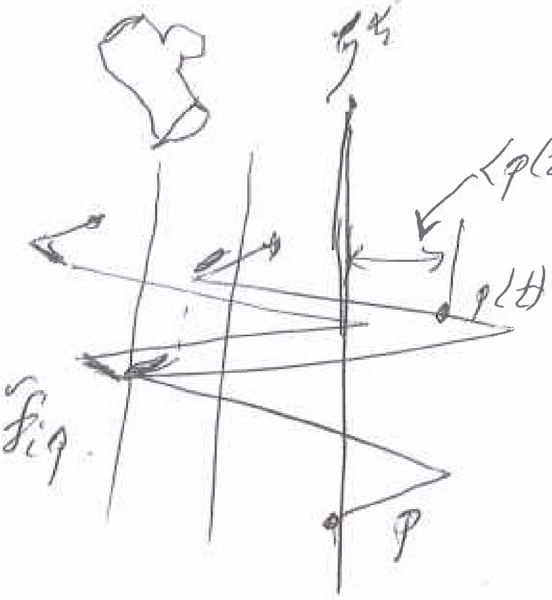
Let $x \in W$. The Demazure module is

$$L(\Lambda)_{\geq x} = (U\mathfrak{b}) \cdot u_{x\Lambda}.$$

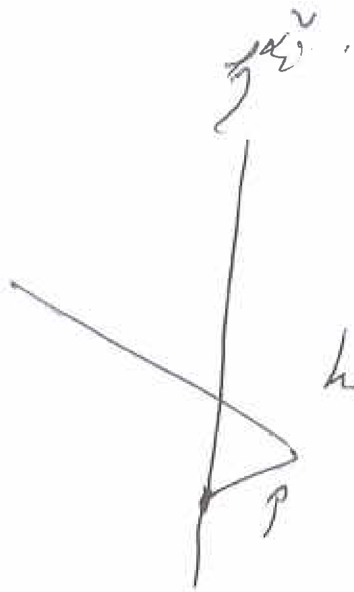
$B(\Lambda)_{\geq 0}$ labels a basis of $L(\Lambda)_{\geq x}$.

Path crystals: paths $p: [0,1] \longrightarrow \mathfrak{h}^*_{\mathbb{R}}$.

root operators: $\tilde{f}_1, \ldots, \tilde{f}_n$.



and



has $\tilde{f}_i p = 0$

Theorem Borel-Bott-Weil.

$$H^0(G/I, \mathcal{L}(\Lambda)) \cong L(\Lambda)$$

$$H^0(\overline{\cancel{I}wI}^{\,0}, \mathcal{L}(\Lambda)) \cong L(\Lambda)_{\geq w}.$$

Theorem Let $D_i = e^{-\rho} \frac{1-s_i}{1-e^{-\alpha_i}} e^{\rho}$. Then

$$\mathrm{char}(L(\Lambda)_{\geq s_i x}) = D_i \,\mathrm{char}(L(\Lambda)_{\geq x}).$$

if $s_i x \geq x$.

Level 0 $\Lambda = \lambda + 0\Lambda_0$ and $\lambda = m_1\omega_1 + \cdots + m_n\omega_n$.

(a) (Periodicity)

$$L(\lambda + 0\Lambda_0) \cong A^{(\lambda)} \otimes L_{loc}(\lambda + 0\Lambda_0)$$

with $A^{(\lambda)} = \mathbb{C}[z_{1,1}, \ldots, z_{1,m_1}]^{S_{m_1}} \otimes \cdots \otimes \mathbb{C}[z_{n,1}, \ldots, z_{n,m_n}]^{S_{m_n}}$

(b) (Level 1 lifting). As $\mathfrak{g}$-modules

$$L_{loc}(\lambda + 0\Lambda_0) \cong L(\Lambda_0)_{\geq t_\lambda}$$

(c) $$\mathrm{char}(L_{loc}(\lambda + 0\Lambda_0)) = E_\lambda(x; q^{-1}, \infty)$$

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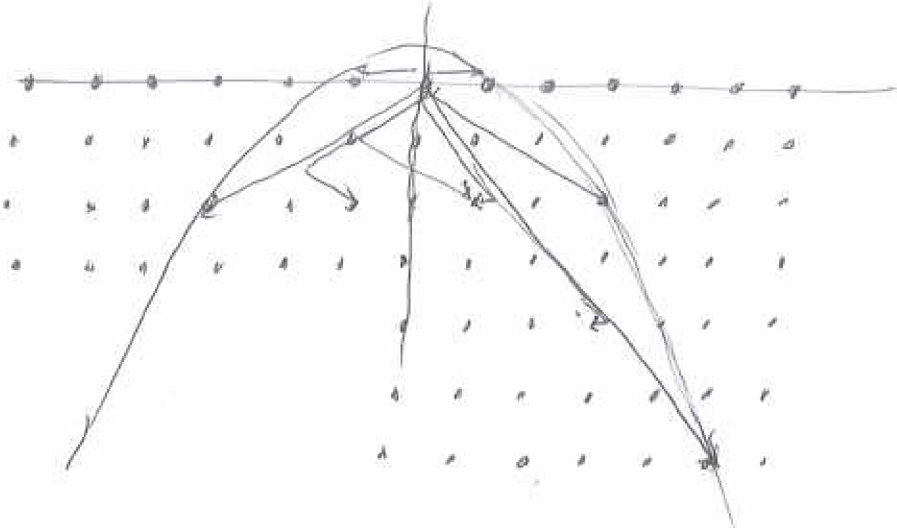
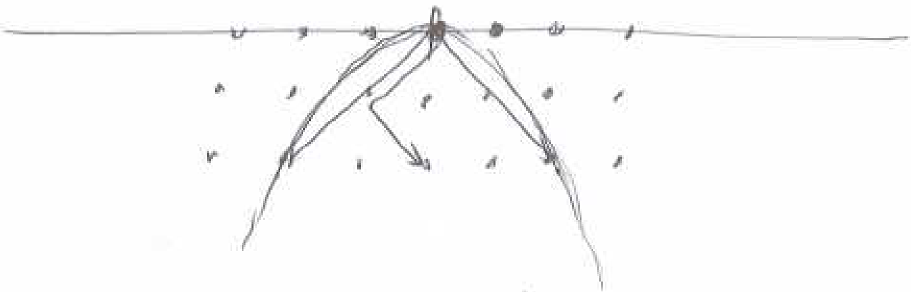
Level 2

$\Lambda = \omega_1 + 2\lambda_0$



Level 1

$\Lambda = \lambda_0$



Level 0

$\Lambda = 2\omega_1 + 0\lambda_0$