W-algebras notes

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1 Introduction

This is mostly lifted from Arakawa's paper [Ar].

2 Vertex algebras, associative filtered algebras and Poisson algebras

2.1 Vertex algebras

A *vertex algebra* is a vector space V with a linear map

$$\begin{array}{rcl} V & \longrightarrow & (\mathrm{End}(V))[[z, z^{-1}]] \\ a & \longmapsto & a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1} \end{array}$$

and elements $\mathbf{1} \in V$ and $T \in \mathrm{End}(V)$ such that

(a) $\mathbf{1}(z) = \mathrm{id}_V$,

(b) If $a, b \in V$ and $a_{(-1)}\mathbf{1} = a$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell}$ then $a_{(n)}b = 0$,

(c) If $a \in V$ then $(Ta)(z) = [T, a(z)] = \frac{d}{dz}a(z)$,

(d) If $a, b \in V$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell}$ then

$$(z-w)^n[a(z), b(w)] = 0, \qquad \text{in } \mathrm{End}(V).$$

A *conformal vertex algebra* is a vertex algebra V with

$$\omega \in V \quad \text{such that} \quad \text{if} \quad \omega(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

then there exists $c_V \in \mathbb{C}$ such that

(e) If $m, n \in \mathbb{Z}$ then $[L_m, L_n] = (m-n)L_{m+n} + \dfrac{(m^3-m)}{12}c_V \delta_{m,-n}$,

(f) $L_{-1} = T$, and

(g) L_0 is diagonalizable on V.

A *graded conformal vertex algebra* is a conformal vertex algebra V with

$$V = \bigoplus_{d\in\frac{1}{2}\mathbb{Z}} V_d, \qquad \text{where} \quad V_d = \{a \in V \mid L_0 a = da\}.$$

Notation:

- The map $V \to \mathrm{End}(V)[[z, z^{-1}]]$ is the *state-field correspondence.*
- A *field* is an element of $\{a(z) \mid a \in V\}$.
- A *mode* is an element of $\{a_{(n)} \mid a \in V, n \in \mathbb{Z}\}$.
- The constant c_V is the *central charge.*
- The degree of a homogenous element $a \in V$ is the *conformal weight of a.*

Let V be a vertex algebra. A V*-module* is a vector space M with a linear map

$$\begin{array}{ccc} V & \longrightarrow & (\mathrm{End}(M))[[z, z^{-1}]] \\ a & \longmapsto & a^M(z) = \sum_{n\in\mathbb{Z}} a^M_{(n)} z^{-n-1} \end{array}$$

such that

(a) $\mathbf{1}^M(z) = \mathrm{id}_M$,

(b) If $a \in V$ and $m \in M$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell}$ then $a^M_{(n)}m = 0$.

(c) If $p, q, r \in \mathbb{Z}$ and $a, b, c \in \mathbb{Z}$ then, in $\mathrm{End}(M)$,

$$\sum_{i \in \mathbb{Z}_{\geq 0}} \binom{p}{i} (a_{(r+i)}b)^M_{(p+q+i)} = \sum_{i \in \mathbb{Z}_{>0}} (-1)^i \binom{r}{i} \big(a^M_{(p+r-i)} b^M_{(q+i)} - (-1)^r b^M_{(q+r+i)} a^M_{(p+i)}\big).$$

Proposition 2.1. *Let V be a vertex algebra.*

(a) The category of V-modules is an abelian category.

(b) V is a V-module (the adjoint module*).*

Proof. Proof idea for (b): Show that if $a, b \in V$ and $p, q, r \in \mathbb{Z}$ then, in $\mathrm{End}(M)$,

$$\sum_{i \in \mathbb{Z}_{\geq 0}} \binom{p}{i} (a_{(r+i)}b)_{(p+q+i)} = \sum_{i \in \mathbb{Z}_{>0}} (-1)^i \binom{r}{i} \big(a_{(p+r-i)} b_{(q+i)} - (-1)^r b_{(q+r+i)} a_{(p+i)}\big).$$

□

Let V be a graded conformal vertex algebra.

- V is *rational*, or *(representation) semisimple*, if every V-module is completely reducible.

2.2 The enveloping algebra $U(V)$ of V

Let V be a graded conformal vertex algebra.

$$V = \bigoplus_{d \in \frac{1}{2}\mathbb{Z}} V_d, \qquad \text{where} \quad V_d = \{a \in V \mid L_0 a = da\}.$$

For homogeneous $a, b \in V$ define

$$a \circ b = \sum_{i \in \mathbb{Z}_{\geq 0}} \binom{\mathrm{wt}(a)}{i} a_{(i-2)}b, \qquad \text{and}$$
$$a * b = \sum_{i \in \mathbb{Z}_{\geq 0}} \binom{\mathrm{wt}(a)}{i} a_{(i-1)}b.$$

The *enveloping algebra of V*, or *(L_0-twisted) Zhu's algebra of V* is

$$U(V) = \frac{V}{O(V)}, \qquad \text{where} \quad O(V) = \mathbb{C}\text{-span}\{a \circ b \mid \text{homogeneous } a, b \in V\},$$

with product

$$\begin{array}{ccc} U(V) \otimes U(V) & \longrightarrow & U(V) \\ (a, b) & \longmapsto & a * b. \end{array}$$

Define a filtration on $U(V)$ by

$$F_d U(V) = (\text{image of } V_{\leq d} \text{ in } U(V)).$$

Proposition 2.2. *The map* $\pi_P\colon \mathrm{Ps(V)} \to \mathrm{gr}_F U(V)$ *given by*

$$\pi_P(a + C_2(V)_p) = (a + (O(V) \cap V_{\le p})) + V_{\le (p-\frac{1}{2})}, \qquad \textit{for } a \in \mathrm{Ps}(V)_p,$$

is a sujective homomorphism of graded Poisson algebras.

Let V be a vertex algebra and let M be a graded V-module. For homogeneous $a \in V$ and $m \in M$ define

$$a \circ m = \sum_{i \in \mathbb{Z}_{\ge 0}} \binom{\mathrm{wt}(a)}{i} a^M_{(i-2)} m.$$

Let

$$O(M) = \mathbb{C}\text{-span}\{a \circ m \mid \text{homogeneous } a \in V \text{ and } m \in M\}.$$

Define

$$U(M) = \frac{M}{O(M)},$$

with $U(V)$-bimodule structure given by

$$a * m = \sum_{i \in \mathbb{Z}_{\ge 0}} \binom{\mathrm{wt}(a)}{i} a^M_{(i-1)} m \qquad \text{and} \qquad m * a = \sum_{i \in \mathbb{Z}_{\ge 0}} \binom{\mathrm{wt}(a) - 1}{i} a^M_{(i-1)} m.$$

Theorem 2.3. *The functor*

$$\begin{array}{ccc} V\text{-}\mathit{Mod} & \longrightarrow & U(V)\text{-}\mathit{biMod} \\ M & \longmapsto & U(M) \end{array} \qquad \textit{is a right exact functor.}$$

2.3 The Poisson algebra $\mathrm{Ps}(V)$ of V

Let V be a graded conformal vertex algebra

$$V = \bigoplus_{d \in \frac{1}{2}\mathbb{Z}} V_d, \qquad \text{where} \quad V_d = \{v \in V \mid L_0 v = dv\},$$

Let

$$C_2(V) = \mathbb{C}\text{-span}\{a_{(-2)} v \mid v \in V\}.$$

The *Poisson algebra of* V, or *Zhu's* C_2*-algebra of* V, is

$$\mathrm{Ps}(V) = \frac{V}{C_2(V)} \qquad \text{with} \quad \bar{a} \cdot \bar{b} = \overline{a_{(-1)} b} \quad \text{and} \quad \{\bar{a}, \bar{b}\} = \overline{a_{(0)} b}.$$

and grading

$$\mathrm{Ps}(V) = \bigoplus_{d \in \frac{1}{2}\mathbb{Z}} \mathrm{Ps}(V)_d, \qquad \text{where} \quad \mathrm{Ps}(V)_d = (\text{image of } V_d \text{ in } \mathrm{Ps}(V)).$$

Proposition 2.4. *Let* V *be a graded conformal vertex algebra. Then* $\mathrm{Ps}(V)$ *is a graded Poisson algebra.*

Let V be a graded conformal vertex algebra.

- V is *finitely strongly generated* if $\mathrm{Ps}(V)$ is a finitely generated ring.
- V is *C_2-cofinite*, or *lisse*, if $\mathrm{Ps}(V)$ is a finite dmensional.

Proposition 2.5. *Let V be a graded conformal vertex algebra and let M be a graded V-module. Define*

$$C_2(M) = \mathbb{C}\text{-}span\{a^M_{(-2)}m \mid a \in V, m \in M\}.$$

Then

$$\mathrm{Ps}(M) = \frac{M}{C_2(M)}, \qquad with \quad \overline{a}\cdot\overline{m} = \overline{a^M_{(-1)}m} \quad and \quad \{\overline{a},\overline{m}\} = \overline{a^M_{(0)}m}.$$

is a Poisson module for $\mathrm{Ps}(V)$*.*

2.4 Poisson algebras and modules

A *Poisson algebra* is a commutative $\mathbb{C}$-algebra R with a bilinear map

$$\begin{matrix} R\otimes R & \longrightarrow & R \\ (r_1,r_2) & \longmapsto & \{r_1,r_2\} \end{matrix} \qquad \text{such that}$$

(a) If $a,b\in R$ then $\{a,b\} = -\{b,a\}$,

(b) If $a,b,c\in R$ then $\{a,\{b,c\}\}+\{b,\{c,a\}\}+\{c,\{a,b\}\} = 0$,

(c) If $a,b,c\in R$ then $\{a,bc\} = \{a,b\}c + b\{a,c\}$.

Let R be a Poisson algebra (really we should use graded Poisson superalgebras). A *Poisson module for R* is an R-module M with a bilinear map

$$\begin{matrix} R\otimes M & \longrightarrow & M \\ (r,m) & \longmapsto & \{r,m\} \end{matrix} \qquad \text{such that}$$

(a) If $r_1,r_2\in R$ then $\{r_1,r_2\}m = r_1r_2m - r_2r_1m$,

(b) If $r_1,r_2\in R$ and $m\in M$ then $\{r_1,r_2m\} = \{r_1,r_2\}m + r_2\{r_1,m\}$.

(c) If $r_1,r_2\in R$ and $M\in M$ then $\{r_1r_2,m\} = r_1\{r_2,m\} + r_2\{r_1,m\}$.

Notation:

- R-PMod is the category of Poisson modules for R.

2.5 Associative filtered algebras

An *associative filtered algebra* is a $\mathbb{C}$-algebra U with a filtration

$$\mathbb{C} = U_0 \subseteq U_1 \subseteq \cdots \quad \text{such that} \quad \left(\bigcup_{i\in\mathbb{Z}_{\geq 0}} U_i \right) = U \quad \text{and} \quad U_iU_j \subseteq U_{i+j}.$$

WHAT WE REALLY NEED IS INCREASING EXHAUSTIVE SEPARATED FILTRATION. WHAT DOES THIS MEAN?? Notation:

- A-biMod is the category of A-bimodules.

Let A be an associative filtered algebra and let $M \in A$-biMod. A *compatible filtration* is a $\frac{1}{2}\mathbb{Z}$-filtration on M such that if $p, q \in \frac{1}{2}\mathbb{Z}$ then

$$(F_pU)\cdot(F_qM) \subseteq F_{p+q}M, \quad (F_qM)\cdot(F_pA) \subseteq F_{p+q}M, \quad \text{and} \quad [F_pA, F_qM] \subseteq F_{p+q-1}M.$$

Proposition 2.6. *Let U be an associative filtered algebra.*

(a) Then

$$\mathrm{gr}_F U = \bigoplus_{p\in\frac{1}{2}\mathbb{Z}} \frac{F_pU}{F_{p-\frac{1}{2}}U} \qquad \textit{with} \quad \{\bar{a}, \bar{b}\} = \overline{ab - ba},$$

is a graded Poisson algebra.

(b) Let $M \in A$-biMod with a compatible filtration. Then

$$\mathrm{gr}_F M = \bigoplus_{p\in\frac{1}{2}\mathbb{Z}} \left(\frac{F_pM}{F_{p-\frac{1}{2}}M} \right) \qquad \textit{is a graded Poisson module for } \mathrm{gr}_F U.$$

3 The vertex algebra $V^k(\mathfrak{g})$

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra with nondegenerate ad-invariant inner product $(\,|\,)\colon \mathfrak{g}\otimes\mathfrak{g} \to \mathbb{C}$. Let

$$\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K, \qquad \text{with} \quad [K, xt^m] = 0, \quad \text{and}$$
$$[xt^m, yt^n] = [x, y]t^{m+n} + m(x|y)\delta_{m,-n}K,$$

for $x, y \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$. Let $k \in \mathbb{C}$. The *universal affine vertex algebra associated to* $\mathfrak{g}$ *at level* k is the vector space

$$V^k(\mathfrak{g}) = U\hat{g} \otimes_{U(\mathfrak{g}[t]\oplus\mathbb{C}K)} \mathbb{C}_k,$$

where $\mathbb{C}_k = \mathbb{C}\text{-span}\{v\}$ with $Kv = kv$ and $xt^mv = 0$ for $x \in \mathfrak{g}$ and $m \in \mathbb{Z}$, and $V^k(\mathfrak{g})$ has vertex algebra structure determined by

$$\mathbf{1} = v, \qquad (xt^{-1}\mathbf{1})(z) = \sum_{n\in\mathbb{Z}} (xt^n)z^{-n-1}, \quad \text{for } x \in \mathfrak{g},$$

and, if $\{X_1, \ldots, X_\ell\}$ is a basis of $\mathfrak{g}$ and $\{X^1, \ldots, X^n\}$ is the dual basis of $\mathfrak{g}$ with respect to $(\,|\,)$ then

$$\omega = \frac{1}{2(k + h^\vee)} \sum_{i=1}^{\ell} (X_it^{-1})(X^it^{-1})\mathbf{1}. \qquad \text{(the Sugawara vector)}$$

3.1 The associative filtered algebra of $V^k(\mathfrak{g})$

Let $U\mathfrak{g}$ be the enveloping algebra of $\mathfrak{g}$. The *PBW filtration on* $U\mathfrak{g}$ is given by

$$f_{-1}U\mathfrak{g} = 0, \quad F_0U\mathfrak{g} = \mathbb{C}, \quad F_pU\mathfrak{g} = \mathfrak{g}\cdot(F_{p-1}U\mathfrak{g}) + (F_{p-1}\mathfrak{g}).$$

The Poincaré-Birkhoff-Witt theorem gives that

$$\mathrm{gr}_F U\mathfrak{g} \cong S(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*].$$

Let f be a nilpotent element of $\mathfrak{g}$. The *Kazhdan filtration of* $U\mathfrak{g}$ *with respect to* f is given by

$$K_pU\mathfrak{g} = \sum_{i-j\le p} F_iU\mathfrak{g}[j], \qquad \text{where} \quad F_pU\mathfrak{g}[j] = \{u\in F_pU\mathfrak{g} \mid \mathrm{ad}(h)(u) = 2ju\}.$$

Proposition 3.1. *Let f be a nilpotent element of $\mathfrak{g}$ and let*

$$K_0U\mathfrak{g} \subseteq K_1U\mathfrak{g} \subseteq K_2U\mathfrak{g} \subseteq \cdots \ \textit{be the Kazhdan filtration of } U\mathfrak{g}$$

with respect to f. Then

$$\mathrm{gr}_K U\mathfrak{g} \cong S(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*].$$

IS THE PBW FILTRATION THE KAZHDAN FILTRATION OF $U\mathfrak{g}$ with respect to the regular nilpotent????

Proposition 3.2. *(Frenkel-Zhu) (equation (25) in Arakawa)*

(a) $U(V^k(\mathfrak{g})) \cong U\mathfrak{g}$.

(b) The filtration on $U(V^k(\mathfrak{g}))$ given by

$$F_p(U(V^k(\mathfrak{g})) = (\textit{image of } V^k(\mathfrak{g})_{\le p} \textit{ in } U(V^k(\mathfrak{g})))$$

corresponds to the PBW??? or Kazhdan??? filtration on $U\mathfrak{g}$.

For a $U\mathfrak{g}$-bimodule M define $\mathrm{ad}\colon \mathfrak{g}\to \mathrm{End}(M)$ by

$$\mathrm{ad}(x)m = xm - mx, \qquad \text{for } x\in\mathfrak{g} \text{ and } m\in M.$$

The action of $\mathfrak{g}$ by ad is the *adjoint action of $\mathfrak{g}$ on M.*

- $U\mathfrak{g}$-biMod is the category of $U\mathfrak{g}$-bimodules.
- $\mathcal{HC}$ is the full subcategory of $U\mathfrak{g}$-biMod consisting of M such that

  the ad action of $\mathfrak{g}$ on M is locally finite.

Proposition 3.3.

(a) $U\colon \mathrm{KL}_k \to \mathcal{HC}$ *is a right exact functor.*

(b) If $M\in \mathrm{KL}_k$ and M is finitely generated then $U(M)$ is a finitely generated $U\mathfrak{g}$ module.

Proposition 3.4.

(a) $V^k(\mathfrak{g}) \in \mathrm{KL}_k^\Delta$.

(b) $\mathrm{KL}_k^\Delta = \{m\in \mathrm{KL}_k \mid M$ *is a free* $U(\mathfrak{g}[t^{-1}]t^{-1})$*-module of finite rank*$\}$.

(c) If $M\in\mathrm{KL}_k^\Delta$ then $\mathrm{Ps}(M) \cong \mathrm{gr}_F U(M)$.

(d) $U\colon \mathrm{KL}_k^\Delta \to \mathcal{HC}$ *is an exact functor.*

3.2 The Poisson algebra of $V^k(\mathfrak{g})$

The commutative algebra $\mathbb{C}[\mathfrak{g}^*] = S(\mathfrak{g})$ is a Poisson algebra with the

Kirillov-Kostant Poisson bracket.

Proposition 3.5.

(a) $C_2(V^k(\mathfrak{g})) = \mathfrak{g}[t^{-1}]t^{-2}V^k(\mathfrak{g})$.

(b) The map

$$\Phi\colon \mathbb{C}[\mathfrak{g}^*] \to \mathrm{Ps}(V^k(\mathfrak{g})) \qquad \text{determined by} \qquad \Phi(x) = \overline{(xt^{-1})\mathbf{1}} \text{ for } x \in \mathfrak{g},$$

is an isomorphism of Poisson algebras.

A $\mathbb{C}[\mathfrak{g}^*]$-*Poisson module* is a $\mathbb{C}[\mathfrak{g}^*]$-module M with a linear map $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{End}(M)$ such that

(a) If $x, y \in \mathfrak{g}$ then $\mathrm{ad}(\{x,y\}) = \mathrm{ad}(x)\mathrm{ad}(y) - \mathrm{ad}(y)\mathrm{ad}(x)$,

(b) If $x \in \mathfrak{g}$, $f \in \mathbb{C}[\mathfrak{g}^*]$ and $m \in M$ then

$$\mathrm{ad}(x)(fm) = \{x, f\}m + f\mathrm{ad}(x)(m).$$

The action of $\mathfrak{g}$ by ad is the *adjoint action of* $\mathfrak{g}$ *on* M.

- $\mathbb{C}[\mathfrak{g}^*]$-PMod is the category of $\mathbb{C}[\mathfrak{g}^*]$-Poisson modules.
- $\overline{\mathcal{HC}}$ is the full subcategory of $\mathbb{C}[\mathfrak{g}^*]$-PMod consisting of M such that

  the ad action of $\mathfrak{g}$ on M is locally finite.

3.3 $V^k(\mathfrak{g})$-modules

A *smooth* $\hat{\mathfrak{g}}$-*module* is a $\hat{\mathfrak{g}}$-module M such that if $m \in M$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that

$$\text{if } n \in \mathbb{Z}_{\geq \ell} \text{ and } x \in \mathfrak{g} \quad \text{then} \quad (xt^n)m = 0.$$

Proposition 3.6. *A* $V^k(\mathfrak{g})$*-module is the same thing as a smooth* $\hat{\mathfrak{g}}$*-module of level* k*.*

View $\mathfrak{g}$ as a Lie subalgebra of $\mathfrak{g}$ by the inclusion $x \mapsto xt^0$.

- $V^k(\mathfrak{g})$-Mod is the abelian category of $V^k(\mathfrak{g})$-modules.
- $V^k(\mathfrak{g})$-gMod is the full subcategory of $V^k(\mathfrak{g})$-Mod of

  positively graded $V^k(\mathfrak{g})$-modules.

- KL_k is the full subcategory of graded $V^k(\mathfrak{g})$-modules M such that

  $\mathfrak{g}$ acts locally finitely on M.

- KL_k^{Δ} is the full subcategory of KL_k-modules M which satisfy: there exists $r \in \mathbb{Z}_{\geq 0}$ and

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_r = M \qquad \text{with} \quad \frac{M_i}{M_{i-1}} \cong \Delta_{\mathfrak{g}}^{\hat{\mathfrak{g}}}(E_i),$$

for finite dimensional $\mathfrak{g}$-modules $E_1, \ldots, E_r$.

Proposition 3.7. *Let M be a $V^k(\mathfrak{g})$-module.*

(a) $C_2(M) = \mathfrak{g}[t^{-1}]t^{-2}M$.

(b) $\mathrm{Ps}(M) = \dfrac{M}{\mathfrak{g}[t^{-1}]t^{-2}M}$ *is a Poisson module for* $\mathbb{C}[\mathfrak{g}^*]$ *with*

$$x \cdot \overline{m} = \overline{(xt^{-1})m} \quad \textit{and} \quad \{x, \overline{m}\} = \overline{(xt^0)m}, \qquad \textit{for } x \in \mathfrak{g} \textit{ and } m \in M.$$

References

[Ar] T. Arakawa, *Rationality of W-algebras: principal nilpotent cases* arXiv:1211.7124.