University of Melbourne
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Flag varieties

$$\begin{array}{ll} G = \text{complex reductive algebraic group} & \mathfrak{g} = \mathrm{Lie}(G) \\ \cup\!\!\mid & \cup\!\!\mid \\ B = \text{Borel subgroup} & \mathfrak{b} = \mathrm{Lie}(B) \\ \cup\!\!\mid & \cup\!\!\mid \\ T = \text{maximal torus} & \mathfrak{h} = \mathrm{Lie}(T) \end{array}$$

Let

$$\mathfrak{h}_{\mathbb{Z}}^* = \mathrm{Hom}(T, \mathbb{C}^\times), \quad \mathfrak{h}_{\mathbb{Z}} = \mathrm{Hom}(\mathbb{C}^\times, T), \quad W_0 = \frac{N(T)}{T}.$$

G/B is the <u>flag variety</u>

Then

$$G = \bigsqcup_{w \in W_0} BwB \quad \text{and} \quad G = \bigsqcup_{w \in W_0} B^- wB$$

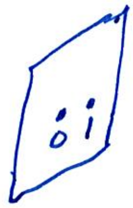
where $B^- = w_0 B w_0^{-1}$ is the opposite Borel ($B \cap B^- = T$).

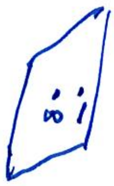
Note that

$$H_2(G/B) \cong \mathrm{Hom}(\mathbb{C}^\times, T) = \mathfrak{h}_{\mathbb{Z}}.$$

<u>The genus 0 curve with 3 marked points</u> A. Ram

$\mathbb{P}^1 =$ 

$= \left\{ \begin{bmatrix}1&0\\c&1\end{bmatrix} \,\middle|\, c\in\mathbb{C}\right\} \cup \left\{\begin{bmatrix}0&1\\1&0\end{bmatrix}\right\} =$  $\cup\ \dot{\infty}$

$= \left\{ \begin{bmatrix}1&0\\0&1\end{bmatrix}\right\} \cup \left\{\begin{bmatrix}c&1\\1&0\end{bmatrix} \,\middle|\, c\in\mathbb{C}\right\} = \dot{0} \cup$ 

with

$$\begin{bmatrix}1&0\\c&1\end{bmatrix} = \begin{bmatrix}c^{-1}&1\\1&0\end{bmatrix}, \quad 0 = \begin{bmatrix}1&0\\0&1\end{bmatrix}, \quad 1 = \begin{bmatrix}1&0\\1&1\end{bmatrix}, \quad \infty = \begin{bmatrix}0&1\\1&0\end{bmatrix}.$$

Note that

$$\mathcal{O}_{\mathbb{P}^1-\{\infty\}} = \mathcal{O}_{\mathbb{C}} = \mathbb{C}[t]$$

$$\mathcal{O}_{\mathbb{P}^1-\{0\}} = \mathcal{O}_{\mathbb{C}} = \mathbb{C}[t^{-1}]$$

$$\mathcal{O}_{\mathbb{P}^1-\{0,\infty\}} = \mathcal{O}_{\mathbb{C}^\times} = \mathbb{C}[t,t^{-1}]$$

<u>Note:</u> In GL_2/B, let $\begin{bmatrix}a&b\\c&d\end{bmatrix} = \begin{pmatrix}a&b\\c&d\end{pmatrix}B$.

$$\begin{bmatrix}1&0\\c&1\end{bmatrix} = \begin{pmatrix}c^{-1}&1\\1&0\end{pmatrix}\begin{pmatrix}c&1\\0&-c^{-1}\end{pmatrix}B = \begin{bmatrix}c^{-1}&1\\1&0\end{bmatrix}$$

The space $M_3 = M_3(G/B)$

$$M_3 = \{c : \mathbb{P}^1 \to G/B\} = \mathrm{Mor}(\mathbb{P}^1, G/B)$$

Now, $H_2(G/B) = \mathfrak{h}_{\mathbb{Z}}$ and $c : \mathbb{P}^1 \to G/B$ gives

$$\begin{array}{rcl} H_*(\mathbb{P}^1) & \xrightarrow{c_*} & H_*(G/B) \\ [\mathbb{P}^1] & \longmapsto & \tau \in H_2(G/B). \end{array}$$

For $\tau \in \mathfrak{h}_{\mathbb{Z}}$ let

$$M_{3,\tau} = \{c \in M_3 \mid c_*([\mathbb{P}^1]) = \tau\}$$

so that

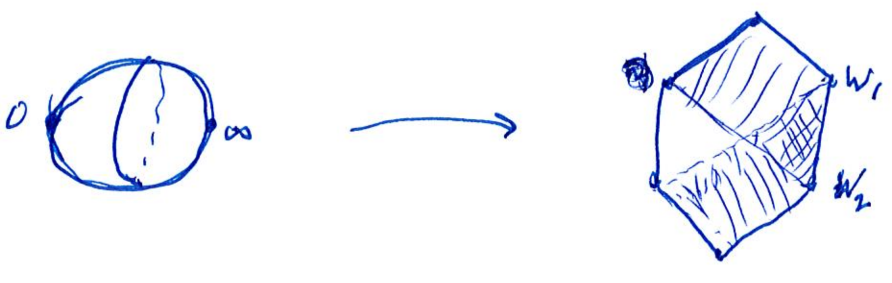
$$M_3 = \bigsqcup_{\tau \in \mathfrak{h}_{\mathbb{Z}}} M_{3,\tau}$$

For $w_1, w_2 \in W_0$ let

$$M_{3,\tau}^{w_1,w_2} = \left\{ c \in M_{3,\tau} \;\middle|\; \begin{array}{l} c(\infty) \in B^- w_1 B \\ c(0) \in B w_2 B \end{array} \right\}$$

so that

$$M_{3,\tau} = \bigsqcup_{w_1, w_2 \in W_0} M_{3,\tau}^{w_1,w_2}$$



Bijections

$$\begin{array}{ccc}
\mathcal{M}_{3,\tau}^{w_1,w_2} & & B^- w_1 B \\
\uparrow\wr & & \updownarrow \\
I^- w_1 HU^+ \cap I w_2 t_\tau HU^+ & & B^- w_1 HU^+ \\
\uparrow & & \updownarrow \\
I^- w_1 t_{\omega p} I \cap I w_2 t_{\tau+\omega p} I & & B^- w_1 I
\end{array}$$

$b_1 b_2 =$ 

$\in I^- x_1 I \cap I x_2 I$

$$x_{\gamma_1}(0) \cdots x_{\gamma_{k-1}}(0)\, x_{\gamma_k}(c_k^{-1})\, x_{\gamma_{k+1}}(c_{k+1}) \cdots x_{\gamma_l}(c_l)\, x_1 b$$

$$= x_{\beta_1}(d_1) \cdots x_{\beta_l}(d_l)\, x_2 b_1 b_2 b \quad \text{with} \quad \begin{array}{l} b_1 \in I \cap (HU^+)^c \\ b_2 \in I \cap HU^+ \end{array}$$

$$= \underbrace{x_{\beta_1}(c_1) \cdots x_{\beta_l}(d_l)(x_2 b_1 x_2^{-1})}\, x_2 b_2$$

$\in I x_2 HU$, since $x_2 b_1 x_2^{-1} \in I$ and $b_2 \in HU^+$.