What is a flag variety?

Arun Ram
University of Melbourne

Victorian Algebra Conference
Sydney 30 November 2015

Outline of this talk

Outline of this talk

Part 1. The flag variety

Outline of this talk

Part 1. The flag variety

Part 2. The flag variety

Outline of this talk

Part 1. The flag variety

Part 2. The flag variety

Part 3. The flag variety

Outline of this talk

Part 1. The flag variety

Part 2. The flag variety

Part 3. The flag variety

Part 4. The flag variety

Outline of this talk

Part 1. The flag variety G/B

Part 2. The flag variety

Part 3. The flag variety

Part 4. The flag variety

Outline of this talk

Part 1. The flag variety G/B

Part 2. The flag variety Fl

Part 3. The flag variety

Part 4. The flag variety

Outline of this talk

Part 1. The flag variety G/B

Part 2. The flag variety Fl

Part 3. The flag variety X

Part 4. The flag variety

Outline of this talk

Part 1. The flag variety G/B

Part 2. The flag variety Fl

Part 3. The flag variety X

Part 4. The flag variety $P(V)$

Outline of this talk

Part 1. The flag variety: Cosets G/B

Part 2. The flag variety Fl

Part 3. The flag variety X

Part 4. The flag variety $P(V)$

Outline of this talk

Part 1. The flag variety: Cosets G/B

Part 2. The flag variety: Flags *Fl*

Part 3. The flag variety X

Part 4. The flag variety *P(V)*

Outline of this talk

Part 1. The flag variety: Cosets G/B

Part 2. The flag variety: Flags Fl

Part 3. The flag variety: The building X

Part 4. The flag variety $P(V)$

Outline of this talk

Part 1. The flag variety: Cosets G/B

Part 2. The flag variety: Flags Fl

Part 3. The flag variety: The building X

Part 4. The flag variety: The lattice $P(V)$

Part 1. The flag variety: Cosets G/B

Part 1. The flag variety: Cosets G/B

$$G = \{3 \times 3 \text{ invertible matrices with entries in } \mathbb{F}\}$$

$$\cup\!\!|$$

$$B = \left\{ \begin{pmatrix} a_1 & c_1 & c_2 \\ 0 & a_2 & c_3 \\ 0 & 0 & a_3 \end{pmatrix} \;\middle|\; \begin{matrix} a_1, a_2, a_3 \in \mathbb{F}^\times \\ c_1, c_2, c_3 \in \mathbb{F} \end{matrix} \right\}$$

The <u>flag variety</u> is

$$G/B = \{ gB \mid g \in G \}$$

Part 1. The flag variety: Cosets G/B

$$G/B = \{ gB \mid g \in G \}$$

Part 1. The flag variety: Cosets G/B

$$G/B = \{ gB \mid g \in G \}$$

$$= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} B,\ \begin{pmatrix} c_1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} B,\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_1 & 1 \\ 0 & 1 & 0 \end{pmatrix} B,\ \begin{pmatrix} c_1 & c_2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} B,\ \begin{pmatrix} c_1 & 1 & 0 \\ c_2 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} B,\ \begin{pmatrix} c_2 & c_3 & 1 \\ c_1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} B \right\}$$

Part 1. The flag variety: Cosets G/B

$$G/B = \{ gB \mid g \in G \}$$

$$= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} B,\ \begin{pmatrix} c_1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} B,\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_1 & 1 \\ 0 & 1 & 0 \end{pmatrix} B,\ \begin{pmatrix} c_1 & c_2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} B,\ \begin{pmatrix} c_1 & 1 & 0 \\ c_2 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} B,\ \begin{pmatrix} c_2 & c_3 & 1 \\ c_1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} B \right\}$$

$$\mathrm{Card}(G) = \mathrm{Card}(G/B)\,\mathrm{Card}(B)$$

Part 1. The flag variety: Cosets G/B

$$G/B = \{ gB \mid g \in G \}$$

$$= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} B \quad \begin{pmatrix} c_1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} B \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_1 & 1 \\ 0 & 1 & 0 \end{pmatrix} B \quad \begin{pmatrix} c_1 & c_2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} B \quad \begin{pmatrix} c_1 & 1 & 0 \\ c_2 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} B \quad \begin{pmatrix} c_2 & c_3 & 1 \\ c_1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} B \right\}$$

$$\mathrm{Card}(G) = \mathrm{Card}(G/B)\,\mathrm{Card}(B)$$

$$= \left(1 + \begin{matrix} \mathrm{Card}(\mathbb{F}) + \mathrm{Card}(\mathbb{F})^2 \\ \mathrm{Card}(\mathbb{F}) + \mathrm{Card}(\mathbb{F})^2 \end{matrix} + \mathrm{Card}(\mathbb{F})^3 \right)$$

$$\cdot\, \mathrm{Card}(\mathbb{F}^\times)^3 \cdot \mathrm{Card}(\mathbb{F})^{3(3-1)/2}$$

Part 1. The flag variety: Cosets G/B

The field $\mathbb{F}$ with 1 element

$$G/B = \{ gB \mid g \in G \}$$

$$= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} B, \quad \begin{pmatrix} c_1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} B, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_1 & 1 \\ 0 & 1 & 0 \end{pmatrix} B, \quad \begin{pmatrix} c_1 & c_2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} B, \quad \begin{pmatrix} c_1 & 1 & 0 \\ c_2 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} B, \quad \begin{pmatrix} c_2 & c_3 & 1 \\ c_1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} B \right\}$$

$$\mathrm{Card}(G) = \mathrm{Card}(G/B)\,\mathrm{Card}(B)$$

$$= \left(1 + \begin{matrix} \mathrm{Card}(\mathbb{F}) + \mathrm{Card}(\mathbb{F})^2 \\ \mathrm{Card}(\mathbb{F}) + \mathrm{Card}(\mathbb{F})^2 \end{matrix} + \mathrm{Card}(\mathbb{F})^3 \right)$$

$$\cdot\, \mathrm{Card}(\mathbb{F}^\times)^3 \cdot \mathrm{Card}(\mathbb{F})^{3(3-1)/2}$$

Part 1. The flag variety: Cosets G/B

The field $\mathbb{F}$ with 1 element

$$G/B = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} B, \quad \begin{pmatrix} c_1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} B, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_1 & 1 \\ 0 & 1 & 0 \end{pmatrix} B, \quad \begin{pmatrix} c_1 & c_2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} B, \quad \begin{pmatrix} c_1 & 1 & 0 \\ c_2 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} B, \quad \begin{pmatrix} c_2 & c_3 & 1 \\ c_1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} B \right\}$$

$\mathbb{F} = \{0\}$, $\quad c_1, c_2, c_3 \in \mathbb{F}$

$$\mathrm{Card}(G) = \mathrm{Card}(G/B)\,\mathrm{Card}(B)$$

$$= \left(1 + \begin{matrix} \mathrm{Card}(\mathbb{F}) + \mathrm{Card}(\mathbb{F})^2 \\ \mathrm{Card}(\mathbb{F}) + \mathrm{Card}(\mathbb{F})^2 \end{matrix} + \mathrm{Card}(\mathbb{F})^3 \right)$$

$$\cdot\, \mathrm{Card}(\mathbb{F}^\times)^3 \cdot \mathrm{Card}(\mathbb{F})^{3(3-1)/2}$$

Part 1. The flag variety: Cosets G/B

The field $\mathbb{F}$ with 1 element

$$G/B = \left\{ \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}B,\ \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix}B,\ \begin{pmatrix} 1&0&0\\0&0&1\\0&1&0 \end{pmatrix}B,\ \begin{pmatrix} 0&0&1\\1&0&0\\0&1&0 \end{pmatrix}B,\ \begin{pmatrix} 0&1&0\\0&0&1\\1&0&0 \end{pmatrix}B,\ \begin{pmatrix} 0&0&1\\0&1&0\\1&0&0 \end{pmatrix}B \right\}$$

$\mathbb{F} = \{0\}$, $\quad c_1, c_2, c_3 \in \mathbb{F}$

$$\mathrm{Card}(G) = \mathrm{Card}(G/B)\,\mathrm{Card}(B)$$

$$= \left(1 + \begin{matrix} \mathrm{Card}(\mathbb{F}) + \mathrm{Card}(\mathbb{F})^2 \\ \mathrm{Card}(\mathbb{F}) + \mathrm{Card}(\mathbb{F})^2 \end{matrix} + \mathrm{Card}(\mathbb{F})^3\right)$$

$$\cdot\, \mathrm{Card}(\mathbb{F}^\times)^3 \cdot \mathrm{Card}(\mathbb{F})^{3(3-1)/2}$$

Part 1. The flag variety: Cosets G/B

The field $\mathbb{F}$ with 1 element

$$G/B = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} B, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} B, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} B, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} B, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} B, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} B \right\}$$

$\mathbb{F} = \{0\}$,

$$B = \left\{ \begin{pmatrix} a_1 & c_1 & c_2 \\ 0 & a_2 & c_3 \\ 0 & 0 & a_3 \end{pmatrix} \;\middle|\; \begin{matrix} a_1, a_2, a_3 \in \mathbb{F}^\times \\ c_1, c_2, c_3 \in \mathbb{F} \end{matrix} \right\}$$

$$\mathrm{Card}(G) = \mathrm{Card}(G/B)\,\mathrm{Card}(B)$$

Part 1. The flag variety: Cosets G/B

The field $\mathbb{F}$ with 1 element

$$G/B = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} B, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} B, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} B, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} B, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} B, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} B \right\}$$

$\mathbb{F} = \{0\}$, $\mathbb{F}^\times = \{1\}$

$$B = \left\{ \begin{pmatrix} a_1 & c_1 & c_2 \\ 0 & a_2 & c_3 \\ 0 & 0 & a_3 \end{pmatrix} \;\middle|\; \begin{matrix} a_1, a_2, a_3 \in \mathbb{F}^\times \\ c_1, c_2, c_3 \in \mathbb{F} \end{matrix} \right\}$$

$$\mathrm{Card}(G) = \mathrm{Card}(G/B)\,\mathrm{Card}(B)$$

Part 1. The flag variety: Cosets G/B

The field $\mathbb{F}$ with 1 element

$$G/B = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} B \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} B \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} B \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} B \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} B \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} B \right\}$$

$\mathbb{F} = \{0\}, \quad \mathbb{F}^\times = \{1\}$

$$B = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \middle| \begin{array}{l} a_1, a_2, a_3 \in \mathbb{F}^\times \\ c_1, c_2, c_3 \in \mathbb{F} \end{array} \right\}$$

$$\mathrm{Card}(G) = \mathrm{Card}(G/B)\,\mathrm{Card}(B)$$

Part 1. The flag variety: Cosets $\quad G/B$ The field $\mathbb{F}$ with 1 element

$$G/B = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} B, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} B, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} B, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} B, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} B, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} B \right\}$$

$\mathbb{F} = \{0\}, \quad \mathbb{F}^\times = \{1\}$

$$B = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; \begin{array}{l} a_1, a_2, a_3 \in \mathbb{F}^\times \\ c_1, c_2, c_3 \in \mathbb{F} \end{array} \right\}$$

$$\mathrm{Card}(G) = \mathrm{Card}(G/B)\,\mathrm{Card}(B)$$

Part 1. The flag variety: Cosets G/B

The field $\mathbb{F}$ with 1 element

$$G/B = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}B,\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}B,\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}B,\ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}B,\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}B,\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}B \right\}$$

$\mathbb{F} = \{0\}$, $\mathbb{F}^\times = \{1\}$

$$B = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; \begin{matrix} a_1, a_2, a_3 \in \mathbb{F}^\times \\ c_1, c_2, c_3 \in \mathbb{F} \end{matrix} \right\} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

$$\mathrm{Card}(G) = \mathrm{Card}(G/B)\,\mathrm{Card}(B)$$

Part 1. The flag variety: Cosets G/B

The field $\mathbb{F}$ with 1 element

$$G/B = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} B,\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} B,\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} B,\ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} B,\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} B,\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} B \right\}$$

$\mathbb{F} = \{0\}, \quad \mathbb{F}^\times = \{1\}$

$$B = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

$$\mathrm{Card}(G) = \mathrm{Card}(G/B)\,\mathrm{Card}(B)$$

Part 1. The flag variety: Cosets G/B

The field $\mathbb{F}$ with 1 element

$$G/B = \left\{ \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}B,\ \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix}B,\ \begin{pmatrix} 1&0&0\\0&0&1\\0&1&0 \end{pmatrix}B,\ \begin{pmatrix} 0&0&1\\1&0&0\\0&1&0 \end{pmatrix}B,\ \begin{pmatrix} 0&1&0\\0&0&1\\1&0&0 \end{pmatrix}B,\ \begin{pmatrix} 0&0&1\\0&1&0\\1&0&0 \end{pmatrix}B \right\}$$

$\mathbb{F} = \{0\}$, $\mathbb{F}^\times = \{1\}$

$$B = \left\{ \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix} \right\} \qquad G = S_3$$

$$\mathrm{Card}(G) = \mathrm{Card}(G/B)\,\mathrm{Card}(B)$$

Part 1. The flag variety: Cosets G/B

The field $\mathbb{F}$ with 1 element

$$G/B = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} B,\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} B,\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} B,\ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} B,\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} B,\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} B \right\}$$

$\mathbb{F} = \{0\}$, $\mathbb{F}^\times = \{1\}$ $\quad B = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$ $\quad G = S_3$

$$\mathrm{Card}(G) = \mathrm{Card}(G/B)\,\mathrm{Card}(B)$$

Part 1. The flag variety: Cosets G/B

The field $\mathbb{F}$ with 1 element

$$G/B = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} B,\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} B,\ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} B,\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} B,\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} B,\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} B \right\}$$

$\mathbb{F} = \{0\}$, $\mathbb{F}^\times = \{1\}$ $\quad B = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$ $\quad G = S_3$

$$\mathrm{Card}(G) = \mathrm{Card}(G/B)\,\mathrm{Card}(B)$$

$$= \left(1 + \begin{matrix} \mathrm{Card}(\mathbb{F}) + \mathrm{Card}(\mathbb{F})^2 \\ \mathrm{Card}(\mathbb{F}) + \mathrm{Card}(\mathbb{F})^2 \end{matrix} + \mathrm{Card}(\mathbb{F})^3 \right)$$

$$\cdot\, \mathrm{Card}(\mathbb{F}^\times)^3 \cdot \mathrm{Card}(\mathbb{F})^{3(3-1)/2}$$

Part 1. The flag variety: Cosets G/B

The field $\mathbb{F}$ with 1 element

$$G/B = \left\{ \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}B, \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}B, \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}B, \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}B, \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}B, \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}B \right\}$$

$\mathbb{F} = \{0\}$, $\mathbb{F}^\times = \{1\}$ $\quad B = \left\{\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right\}$ $\quad G = S_3$

$$\mathrm{Card}(G) = \mathrm{Card}(G/B)\,\mathrm{Card}(B)$$

$$= \left(1 + \begin{matrix}1 \\ 1\end{matrix} \begin{matrix}+ \\ +\end{matrix} \begin{matrix}1^2 \\ 1^2\end{matrix} + 1^3\right) \cdot \mathrm{Card}(\mathbb{F}^\times)^3 \cdot 1^{3(3-1)/2}$$

Part 1. The flag variety: Cosets G/B

The field $\mathbb{F}$ with 1 element

$$G/B = \left\{ \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix} B \quad \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix} B \quad \begin{pmatrix} 0&0&1\\1&0&0\\0&1&0 \end{pmatrix} B \quad \begin{pmatrix} 1&0&0\\0&0&1\\0&1&0 \end{pmatrix} B \quad \begin{pmatrix} 0&1&0\\0&0&1\\1&0&0 \end{pmatrix} B \quad \begin{pmatrix} 0&0&1\\0&1&0\\1&0&0 \end{pmatrix} B \right\}$$

$\mathbb{F} = \{0\}$, $\quad \mathbb{F}^\times = \{1\}$ $\quad B = \left\{ \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix} \right\}$ $\quad G = S_3$

$$\mathrm{Card}(G) = \mathrm{Card}(G/B)\,\mathrm{Card}(B)$$

$$= \left(1 + \begin{matrix} 1 & + & 1^2 & + & 1^3 \\ 1 & + & 1^2 & & \end{matrix} \right) \cdot 1^3 \cdot 1^{3(3-1)/2}$$

Part 1. The flag variety: Cosets G/B

The field $\mathbb{F}$ with 1 element

$$G/B = \left\{ \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix} B \quad \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix} B \quad \begin{pmatrix} 0&0&1\\1&0&0\\0&1&0 \end{pmatrix} B \quad \begin{pmatrix} 1&0&0\\0&0&1\\0&1&0 \end{pmatrix} B \quad \begin{pmatrix} 0&1&0\\0&0&1\\1&0&0 \end{pmatrix} B \quad \begin{pmatrix} 0&0&1\\0&1&0\\1&0&0 \end{pmatrix} B \right\}$$

$\mathbb{F} = \{0\}$, $\mathbb{F}^\times = \{1\}$ $\quad B = \left\{ \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix} \right\}$ $\quad G = S_3$

$$\mathrm{Card}(G) = \mathrm{Card}(G/B)\,\mathrm{Card}(B) = 6 \cdot 1$$

Part 1. The flag variety: Cosets G/B

The field $\mathbb{F}$ with 1 element

$$G/B = \left\{ \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}B \quad \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix}B \quad \begin{pmatrix} 0&0&1\\1&0&0\\0&1&0 \end{pmatrix}B \quad \begin{pmatrix} 1&0&0\\0&0&1\\0&1&0 \end{pmatrix}B \quad \begin{pmatrix} 0&1&0\\0&0&1\\1&0&0 \end{pmatrix}B \quad \begin{pmatrix} 0&0&1\\0&1&0\\1&0&0 \end{pmatrix}B \right\}$$

$\mathbb{F} = \{0\}$, $\mathbb{F}^\times = \{1\}$ $\quad B = \left\{ \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix} \right\}$ $\quad G = S_3$

$$\mathrm{Card}(G) = \mathrm{Card}(G/B)\,\mathrm{Card}(B) = 6 \cdot 1$$

Part 2. The flag variety: Flags Fl

Part 2. The flag variety: Flags *Fl*

The $\mathbb{F}$-vector space $V = \mathbb{F}^n$

Part 2. The flag variety: Flags Fl

The $\mathbb{F}$-vector space $V = \mathbb{F}^n$ has

basis $\{e_1, e_2, \ldots, e_n\}$, where $e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ ith

Part 2. The flag variety: Flags *Fl*

The $\mathbb{F}$-vector space $V = \mathbb{F}^n$ has

basis $\{e_1, e_2, \ldots, e_n\}$, where $e_i = \begin{array}{cl} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} & \begin{array}{l} \\ \\ \\ i\text{th} \\ \\ \\ \\ \end{array} \end{array}$

$$Fl = \left\{ (0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V) \;\middle|\; \begin{array}{l} V_i \text{ is a subspace} \\ V_i / V_{i-1} \cong \mathbb{F} \end{array} \right\}$$

Part 2. The flag variety: Flags Fl

The $\mathbb{F}$-vector space $V = \mathbb{F}^n$ has

basis $\{e_1, e_2, \ldots, e_n\}$, where $e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ i^{th}

$$Fl = \left\{ (0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V) \;\middle|\; \begin{array}{l} V_i \text{ is a subspace} \\ V_i/V_{i-1} \cong \mathbb{F} \end{array} \right\}$$

Letting $\langle S \rangle = \mathrm{span}(S)$, the favourite flag is

$$F_0 = (0 \subseteq \langle e_1 \rangle \subseteq \langle e_1, e_2 \rangle \subseteq \cdots \subseteq \langle e_1, \ldots, e_n \rangle = V)$$

Part 2. The flag variety: Flags Fl

$$Fl = \left\{ (0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V) \;\middle|\; \begin{array}{l} V_i \text{ is a subspace} \\ V_i / V_{i-1} \simeq \mathbb{F} \end{array} \right\}$$

Part 2. The flag variety: Flags *Fl*

$$Fl = \left\{ (0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V) \;\middle|\; \begin{array}{l} V_i \text{ is a subspace} \\ V_i/V_{i-1} \cong \mathbb{F} \end{array} \right\}$$

Letting $\langle S \rangle = \mathrm{span}(S)$, the favourite flag is

$$F_0 = (0 \subseteq \langle e_1 \rangle \subseteq \langle e_1, e_2 \rangle \subseteq \cdots \subseteq \langle e_1, \ldots, e_n \rangle = V)$$

Part 2. The flag variety: Flags *Fl*

$$Fl = \left\{ (0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V) \;\middle|\; \begin{array}{l} V_i \text{ is a subspace} \\ V_i/V_{i-1} \simeq \mathbb{F} \end{array} \right\}$$

Letting $\langle S \rangle = \mathrm{span}(S)$, the favourite flag is

$$F_0 = (0 \subseteq \langle e_1 \rangle \subseteq \langle e_1, e_2 \rangle \subseteq \cdots \subseteq \langle e_1, \ldots, e_n \rangle = V)$$

$G = GL_n(\mathbb{F})$ acts on $\mathbb{F}^n$:

Part 2. The flag variety: Flags *Fl*

$$Fl = \left\{ (0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V) \;\middle|\; \begin{array}{l} V_i \text{ is a subspace} \\ V_i/V_{i-1} \simeq \mathbb{F} \end{array} \right\}$$

Letting $\langle S \rangle = \mathrm{span}(S)$, the favourite flag is

$$F_0 = (0 \subseteq \langle e_1 \rangle \subseteq \langle e_1, e_2 \rangle \subseteq \cdots \subseteq \langle e_1, \ldots, e_n \rangle = V)$$

$G = GL_n(\mathbb{F})$ acts on $\mathbb{F}^n$:

$$Fl = G \cdot F_0$$

Part 2. The flag variety: Flags *Fl*

$$Fl = \left\{ (0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V) \;\middle|\; \begin{array}{l} V_i \text{ is a subspace} \\ V_i/V_{i-1} \simeq \mathbb{F} \end{array} \right\}$$

Letting $\langle S \rangle = \mathrm{span}(S)$, the favourite flag is

$$F_0 = (0 \subseteq \langle e_1 \rangle \subseteq \langle e_1, e_2 \rangle \subseteq \cdots \subseteq \langle e_1, \ldots, e_n \rangle = V)$$

$G = GL_n(\mathbb{F})$ acts on $\mathbb{F}^n$:

$Fl = G \cdot F_0$ and $\mathrm{Stab}(F_0) = B$

Part 2. The flag variety: Flags *Fl*

$$Fl = \left\{ (0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V) \;\middle|\; \begin{array}{l} V_i \text{ is a subspace} \\ V_i/V_{i-1} \cong \mathbb{F} \end{array} \right\}$$

Letting $\langle S \rangle = \mathrm{span}(S)$, the favourite flag is

$$F_0 = (0 \subseteq \langle e_1 \rangle \subseteq \langle e_1, e_2 \rangle \subseteq \cdots \subseteq \langle e_1, \ldots, e_n \rangle = V)$$

$G = GL_n(\mathbb{F})$ acts on $\mathbb{F}^n$: (orbit/stabilizer)

$Fl = G \cdot F_0$ and $\mathrm{Stab}(F_0) = B$

Part 2. The flag variety: Flags *Fl*

$$Fl = \left\{ (0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V) \;\middle|\; \begin{array}{l} V_i \text{ is a subspace} \\ V_i/V_{i-1} \simeq \mathbb{F} \end{array} \right\}$$

Letting $\langle S \rangle = \mathrm{span}(S)$, the favourite flag is

$$F_0 = (0 \subseteq \langle e_1 \rangle \subseteq \langle e_1, e_2 \rangle \subseteq \cdots \subseteq \langle e_1, \ldots, e_n \rangle = V)$$

$G = GL_n(\mathbb{F})$ acts on $\mathbb{F}^n$: (orbit/stabilizer)

$$Fl = G \cdot F_0 \qquad \text{and} \qquad \mathrm{Stab}(F_0) = B$$

$$\begin{array}{ccc} G/B & \xrightarrow{\sim} & Fl \\ gB & \longmapsto & gF_0 \end{array}$$

Part 2. The flag variety: Flags *Fl*

$$Fl = \left\{ (0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V) \;\middle|\; \begin{array}{l} V_i \text{ is a subspace} \\ V_i/V_{i-1} \simeq \mathbb{F} \end{array} \right\}$$

Letting $\langle S \rangle = \operatorname{span}(S)$, the favourite flag is

$$F_0 = (0 \subseteq \langle e_1 \rangle \subseteq \langle e_1, e_2 \rangle \subseteq \cdots \subseteq \langle e_1, \ldots, e_n \rangle = V)$$

$G = GL_n(\mathbb{F})$ acts on $\mathbb{F}^n$: (orbit/stabilizer)

$$Fl = G \cdot F_0 \qquad \text{and} \qquad \operatorname{Stab}(F_0) = B$$

$$\begin{array}{ccc} G/B & \xrightarrow{\sim} & Fl \\ gB & \longmapsto & gF_0 \end{array}$$

Part 2. The flag variety: Flags *Fl*

The $\mathbb{F}$-vector space $V = \mathbb{F}^n$

$$Fl = \left\{ (0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V) \;\middle|\; \begin{array}{l} V_i \text{ is a subspace} \\ V_i/V_{i-1} \simeq \mathbb{F} \end{array} \right\}$$

Part 2. The flag variety: Flags *Fl*

The $\mathbb{F}$-vector space $V = \mathbb{F}^n$

$$Fl = \left\{ (0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V) \;\middle|\; \begin{array}{l} V_i \text{ is a subspace} \\ V_i/V_{i-1} \cong \mathbb{F} \end{array} \right\}$$

Let M be an R-module.

Part 2. The flag variety: Flags *Fl*

The $\mathbb{F}$-vector space $V = \mathbb{F}^n$

$$Fl = \left\{ (0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V) \;\middle|\; \begin{array}{l} V_i \text{ is a subspace} \\ V_i / V_{i-1} \simeq \mathbb{F} \end{array} \right\}$$

Let M be an R-module.

$$Fl(M) = \left\{ (0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq M) \;\middle|\; \begin{array}{l} V_i \text{ is a submodule} \\ V_i / V_{i-1} \simeq \mathbb{F} \end{array} \right\}$$

Part 2. The flag variety: Flags *Fl*

The $\mathbb{F}$-vector space $V = \mathbb{F}^n$

$$Fl = \left\{ (0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V) \;\middle|\; \begin{array}{l} V_i \text{ is a subspace} \\ V_i/V_{i-1} \simeq \mathbb{F} \end{array} \right\}$$

Let M be an R-module.

$$Fl(M) = \left\{ (0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq M) \;\middle|\; \begin{array}{l} V_i \text{ is a submodule} \\ V_i/V_{i-1} \text{ is simple} \end{array} \right\}$$

Part 2. The flag variety: Flags *Fl*

The $\mathbb{F}$-vector space $V = \mathbb{F}^n$

$$Fl = \left\{ (0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V) \;\middle|\; \begin{array}{l} V_i \text{ is a subspace} \\ V_i/V_{i-1} \simeq \mathbb{F} \end{array} \right\}$$

Let M be an R-module.

$$Fl(M) = \left\{ (0 \subseteq \cdots \subseteq M^2 \subseteq M^1 \subseteq M) \;\middle|\; \begin{array}{l} V_i \text{ is a submodule} \\ V_i/V_{i-1} \text{ is simple} \end{array} \right\}$$

Part 2. The flag variety: Flags *Fl*

The $\mathbb{F}$-vector space $V = \mathbb{F}^n$

$$Fl = \left\{ (0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V) \;\middle|\; \begin{array}{l} V_i \text{ is a subspace} \\ V_i / V_{i-1} \simeq \mathbb{F} \end{array} \right\}$$

Let M be an R-module.

$$Fl(M) = \left\{ (0 \subseteq \cdots \subseteq M^2 \subseteq M^1 \subseteq M) \;\middle|\; \begin{array}{l} V_i \text{ is a submodule} \\ M^i / M^{i+1} \text{ is simple} \end{array} \right\}$$

Part 2. The flag variety: Flags *Fl*

The $\mathbb{F}$-vector space $V = \mathbb{F}^n$

$$Fl = \left\{ (0 \subseteq \cdots \subseteq V^2 \subseteq V^1 \subseteq V) \;\middle|\; \begin{array}{l} V_i \text{ is a subspace} \\ V_i / V_{i-1} \simeq \mathbb{F} \end{array} \right\}$$

Let M be an R-module.

$$Fl(M) = \left\{ (0 \subseteq \cdots \subseteq M^2 \subseteq M^1 \subseteq M) \;\middle|\; \begin{array}{l} V_i \text{ is a submodule} \\ M^i / M^{i+1} \text{ is simple} \end{array} \right\}$$

Part 2. The flag variety: Flags *Fl*

The $\mathbb{F}$-vector space $V = \mathbb{F}^n$

$$Fl = \left\{ (0 \subseteq \cdots \subseteq V^2 \subseteq V^1 \subseteq V) \;\middle|\; \begin{array}{l} V^i \text{ is a subspace} \\ V^i/V^{i+1} \cong \mathbb{F} \end{array} \right\}$$

Let M be an R-module.

$$Fl(M) = \left\{ (0 \subseteq \cdots \subseteq M^2 \subseteq M^1 \subseteq M) \;\middle|\; \begin{array}{l} V_i \text{ is a submodule} \\ M^i/M^{i+1} \text{ is simple} \end{array} \right\}$$

Part 2. The flag variety: Flags *Fl*

The flag variety is

$$Fl = \left\{ (0 \subseteq \cdots \subseteq V^2 \subseteq V^1 \subseteq V) \;\middle|\; \begin{matrix} V^i \text{ is a subspace} \\ V^i/V^{i+1} \simeq \mathbb{F} \end{matrix} \right\}$$

Let M be an R-module.

The space of composition series of M

is

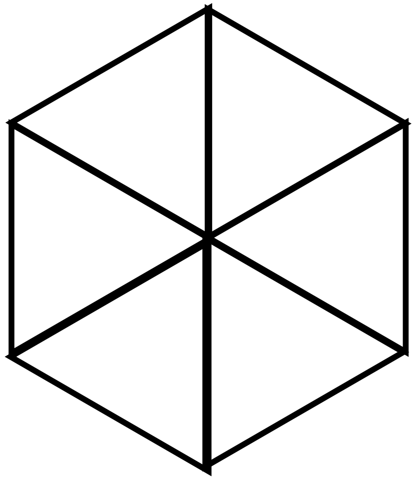
$$Fl(M) = \left\{ (0 \subseteq \cdots \subseteq M^2 \subseteq M^1 \subseteq M) \;\middle|\; \begin{matrix} V_i \text{ is a submodule} \\ M^i/M^{i+1} \text{ is simple} \end{matrix} \right\}$$

Part 3. The flag variety: The building X

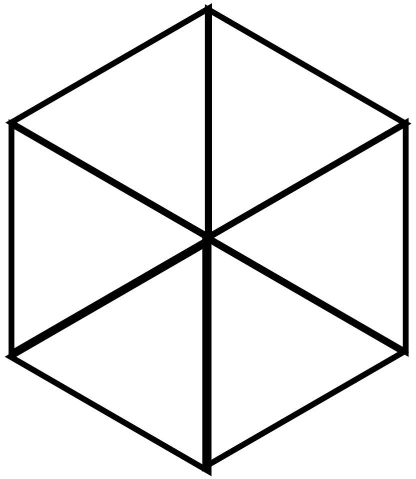
Part 3. The flag variety: The building X "Baby" case:

Part 3. The flag variety: The building X "Baby" case:

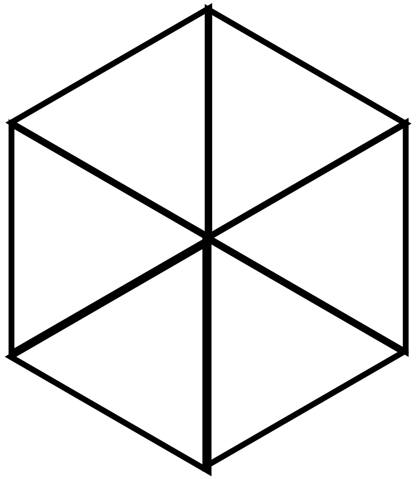
$X =$



Part 3. The flag variety: The building X "Baby" case:

$X =$  $= \{ 1, \ s_1, \ s_1 s_2, \ s_2, \ s_2 s_1, \ s_2 s_1 s_2 \}$

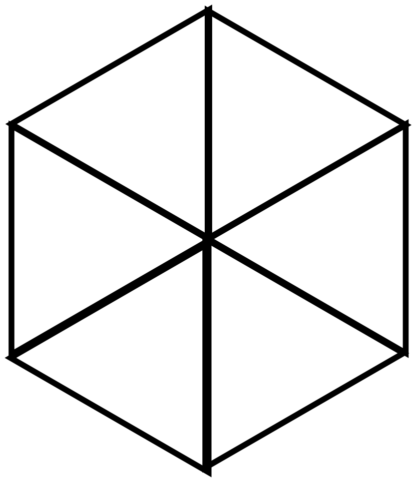
Part 3. The flag variety: The building X "Baby" case:

$X =$  $= \left\{ 1, \quad \begin{matrix} s_1, & s_1 s_2, & \\ & & s_2 s_1 s_2 \\ s_2, & s_2 s_1, & \end{matrix} \right\}$

with $s_2 s_1 s_2 = s_1 s_2 s_1$

Part 3. The flag variety: The building X "Baby" case:

$X =$

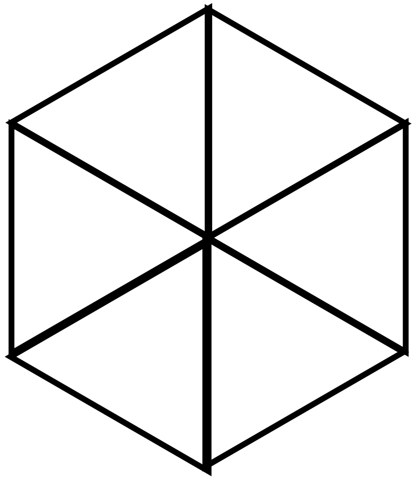


$$= \left\{ 1, \quad s_1, \quad s_1 s_2, \quad s_2, \quad s_2 s_1, \quad s_2 s_1 s_2 \right\}$$

with $s_2 s_1 s_2 = s_1 s_2 s_1$

$$= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\}$$

Part 3. The flag variety: The building X "Baby" case:

$X =$  $= \{ 1,\ s_1,\ s_1 s_2,\ s_2 s_1 s_2,\ s_2,\ s_2 s_1 \}$

with $s_2 s_1 s_2 = s_1 s_2 s_1$

$$= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\} = G/B$$

$G = GL_3(\mathbb{F}_1) = S_3 \qquad B = \{1\}$

Part 3. The flag variety: The building X General case:

$$G/B = \{ gB \mid g \in G \}$$

$$= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} B \quad \begin{matrix} \begin{pmatrix} c_1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} B \\ \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_1 & 1 \\ 0 & 1 & 0 \end{pmatrix} B \end{matrix} \quad \begin{matrix} \begin{pmatrix} c_1 & c_2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} B \\ \\ \begin{pmatrix} c_1 & 1 & 0 \\ c_2 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} B \end{matrix} \quad \begin{pmatrix} c_2 & c_3 & 1 \\ c_1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} B \right\}$$

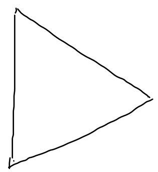
with $c_1, c_2, c_3 \in \mathbb{F}$

Part 3. The flag variety: The building X General case:

$$G/B = \{ gB \mid g \in G \}$$

$$= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} B \quad \begin{pmatrix} c_1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} B \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_1 & 1 \\ 0 & 1 & 0 \end{pmatrix} B \quad \begin{pmatrix} c_1 & c_2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} B \quad \begin{pmatrix} c_1 & 1 & 0 \\ c_2 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} B \quad \begin{pmatrix} c_2 & c_3 & 1 \\ c_1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} B \right\}$$

with $c_1, c_2, c_3 \in \mathbb{F}$

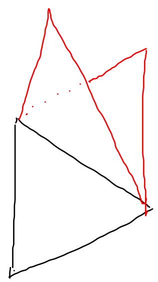
$=$ 

Part 3. The flag variety: The building X General case:

$$G/\mathcal{B} = \{ g\mathcal{B} \mid g \in G \}$$

$$= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\mathcal{B},\ \begin{pmatrix} c_1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\mathcal{B},\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_1 & 1 \\ 0 & 1 & 0 \end{pmatrix}\mathcal{B},\ \begin{pmatrix} c_1 & c_2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\mathcal{B},\ \begin{pmatrix} c_1 & 1 & 0 \\ c_2 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\mathcal{B},\ \begin{pmatrix} c_2 & c_3 & 1 \\ c_1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\mathcal{B} \right\}$$

with $c_1, c_2, c_3 \in \mathbb{F}$

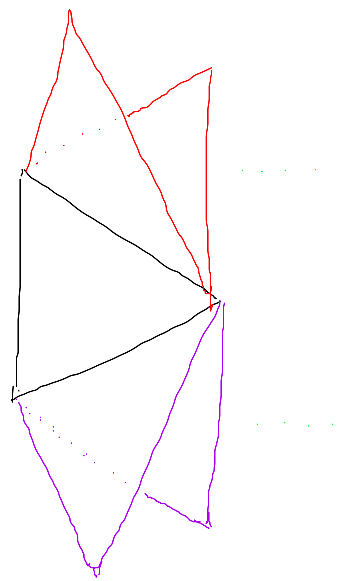
$=$ 

Part 3. The flag variety: The building X General case:

$$G/B = \{ gB \mid g \in G \}$$

$$= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} B \quad \begin{matrix} \begin{pmatrix} c_1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} B \\ \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_1 & 1 \\ 0 & 1 & 0 \end{pmatrix} B \end{matrix} \quad \begin{matrix} \begin{pmatrix} c_1 & c_2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} B \\ \\ \begin{pmatrix} c_1 & 1 & 0 \\ c_2 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} B \end{matrix} \quad \begin{pmatrix} c_2 & c_3 & 1 \\ c_1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} B \right\}$$

with $c_1, c_2, c_3 \in \mathbb{F}$

$=$ 

Part 3. The flag variety: The building $\mathbf{X}$ General case:

$$G/B = \{ gB \mid g \in G \}$$

$$= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} B \quad \begin{pmatrix} c_1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} B \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_1 & 1 \\ 0 & 1 & 0 \end{pmatrix} B \quad \begin{pmatrix} c_1 & c_2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} B \quad \begin{pmatrix} c_1 & 1 & 0 \\ c_2 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} B \quad \begin{pmatrix} c_2 & c_3 & 1 \\ c_1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} B \right\}$$

with $c_1, c_2, c_3 \in \mathbb{F}$

$$= \left\{ B \quad \begin{matrix} f_1(c_1)B, & f_1(c_1)f_2(c_2)B, \\ f_2(c_1)B & f_2(c_1)f_1(c_2)B, \end{matrix} \quad f_2(c_1)f_1(c_2)f_1(c_3)B \right\}$$

with $f_1(c_1)f_2(c_2)f_1(c_1)B = f_2(c_3)f_1(c_1c_3+c_2)f_2(c_1)B$

Part 4. The flag variety: The lattice $P(V)$

Part 4. The flag variety: The lattice $P(V)$

The $\mathbb{F}$-vector space $V = \mathbb{F}^n$

$$\mathcal{Fl} = \left\{ (0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V) \;\middle|\; \begin{array}{l} V_i \text{ is a subspace} \\ V_i / V_{i-1} \simeq \mathbb{F} \end{array} \right\}$$

Part 4. The flag variety: The lattice $P(V)$

The $\mathbb{F}$-vector space $V = \mathbb{F}^n$

$$Fl = \left\{ (0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V) \;\middle|\; \begin{array}{l} V_i \text{ is a subspace} \\ V_i / V_{i-1} \cong \mathbb{F} \end{array} \right\} = G/B$$

Part 4. The flag variety: The lattice $P(V)$

The $\mathbb{F}$-vector space $V = \mathbb{F}^n$

$$\mathcal{F}\ell = \left\{ (0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V) \;\middle|\; \begin{array}{l} V_i \text{ is a subspace} \\ V_i / V_{i-1} \cong \mathbb{F} \end{array} \right\} = G/B$$

$$= \left\{ \begin{array}{l} \text{maximal chains in} \\ \quad \text{the lattice } P(V) \end{array} \right\}$$

Part 4. The flag variety: The lattice $P(V)$

The $\mathbb{F}$-vector space $V = \mathbb{F}^n$

$$Fl = \left\{ (0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V) \;\middle|\; \begin{array}{l} V_i \text{ is a subspace} \\ V_i/V_{i-1} \simeq \mathbb{F} \end{array} \right\} = G/B$$

$$= \left\{ \begin{array}{c} \text{maximal chains in} \\ \text{the lattice } P(V) \end{array} \right\}$$ where

$P(V)$ is the set of subspaces of V

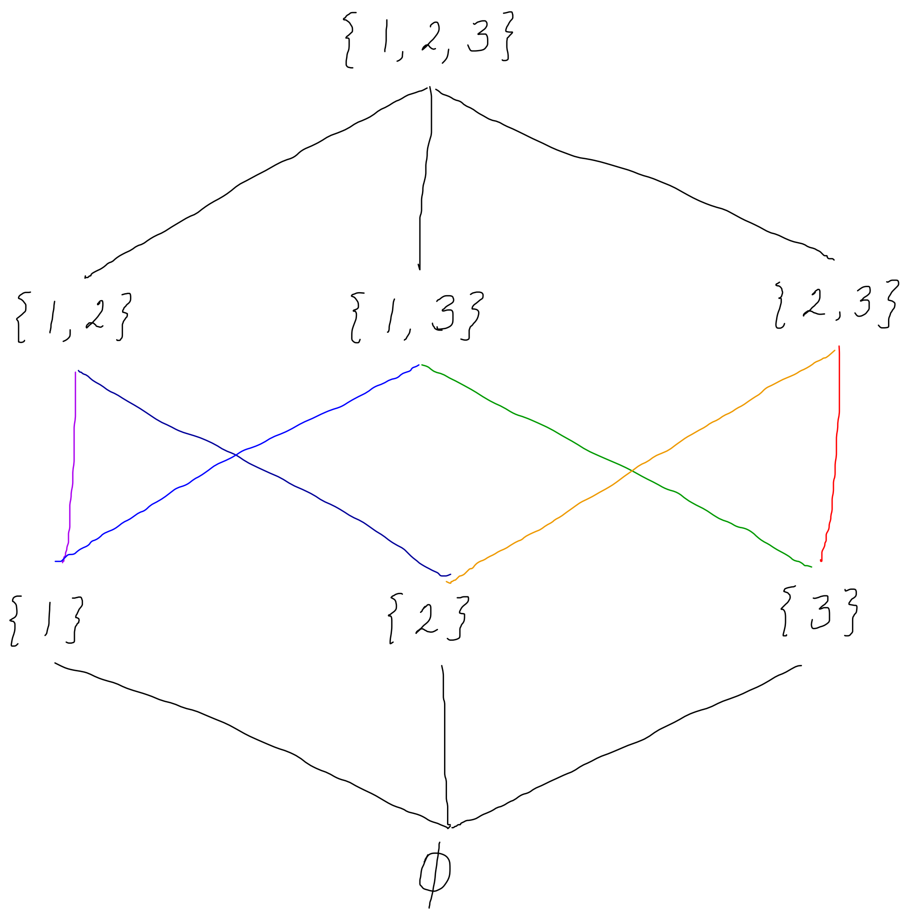
partially ordered by inclusion.

Part 4. The flag variety: The lattice *P(V)* "Baby" case:

The field with one element

$G = GL_3(\mathbb{F}_1) = S_3$ $\qquad$ $B = \{1\}$

G/B is maximal chains in

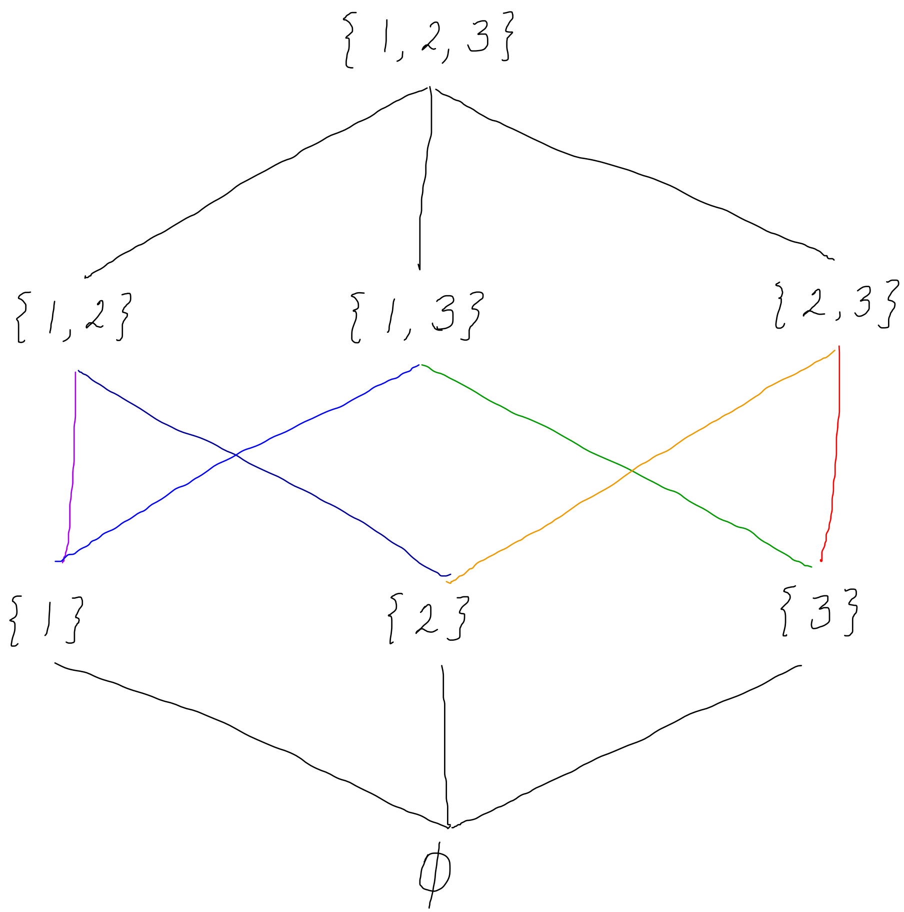


Part 4. The flag variety: The lattice *P(V)* "Baby" case:

The field with one element

$G = GL_3(\mathbb{F}_1) = S_3$ $\quad$ $B = \{1\}$

G/B is maximal chains in the Boolean lattice

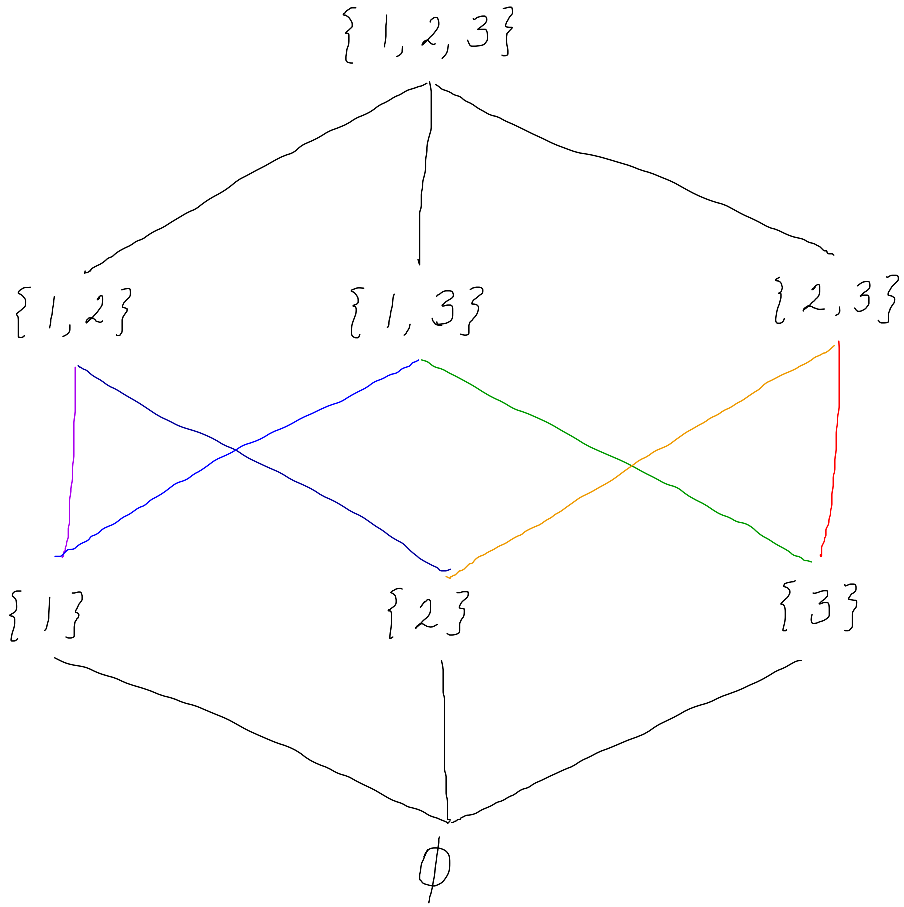


Part 4. The flag variety: The lattice *P(V)* Not baby case:

The field with two elements

$G = GL_3(\mathbb{F}_1) = S_3$ $\qquad$ $B = \{1\}$

G/B is maximal chains in the Boolean lattice



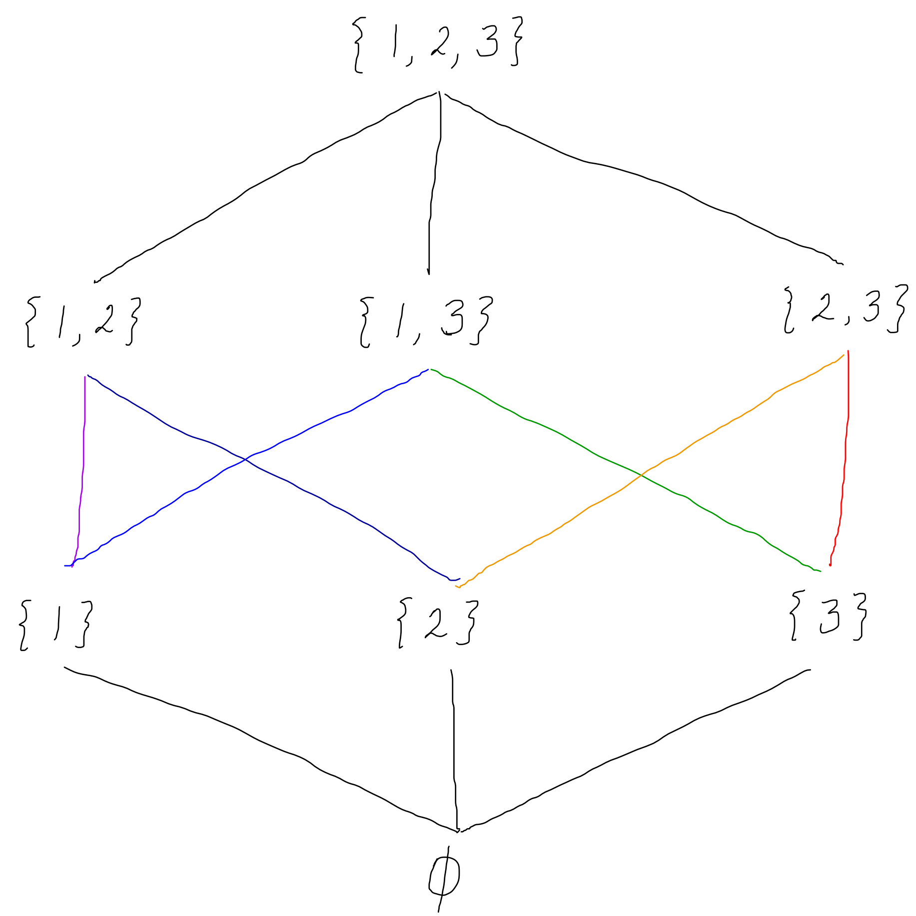
Part 4. The flag variety: The lattice *P(V)* Not baby case:

The field with two elements

$G = GL_3(\mathbb{F}_2)$

$$\mathcal{B} = \left\{ \begin{pmatrix} a_1 & c_1 & c_2 \\ 0 & a_2 & c_3 \\ 0 & 0 & a_3 \end{pmatrix} \;\middle|\; \begin{array}{l} c_1, c_2, c_3 \in \mathbb{F}_2 \\ a_1, a_2, a_3 \in \mathbb{F}_2^\times \end{array} \right\}$$

$G/\mathcal{B}$ is maximal chains in the Boolean lattice



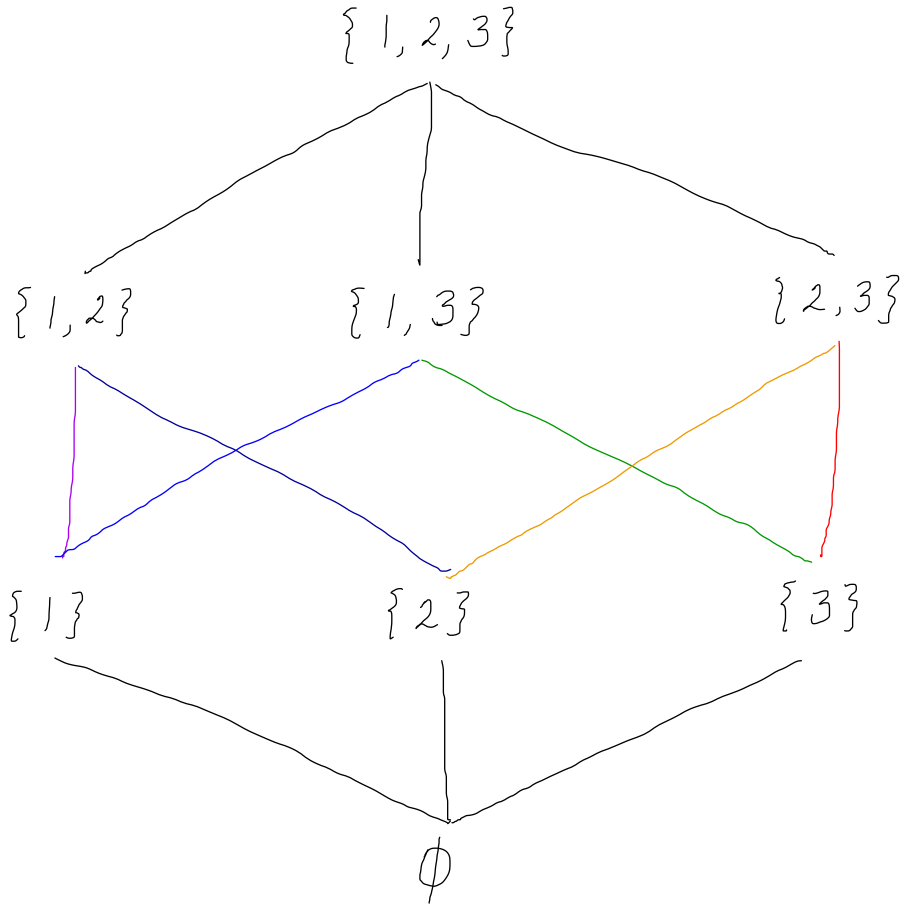
Part 4. The flag variety: The lattice *P(V)* Not baby case:

The field with two elements

$G = GL_3(\mathbb{F}_2)$

$$B = \left\{ \begin{pmatrix} a_1 & c_1 & c_2 \\ 0 & a_2 & c_3 \\ 0 & 0 & a_3 \end{pmatrix} \middle| \begin{array}{l} c_1, c_2, c_3 \in \mathbb{F}_2 \\ a_1, a_2, a_3 \in \mathbb{F}_2^\times \end{array} \right\}$$

G/B is maximal chains in the Fano plane

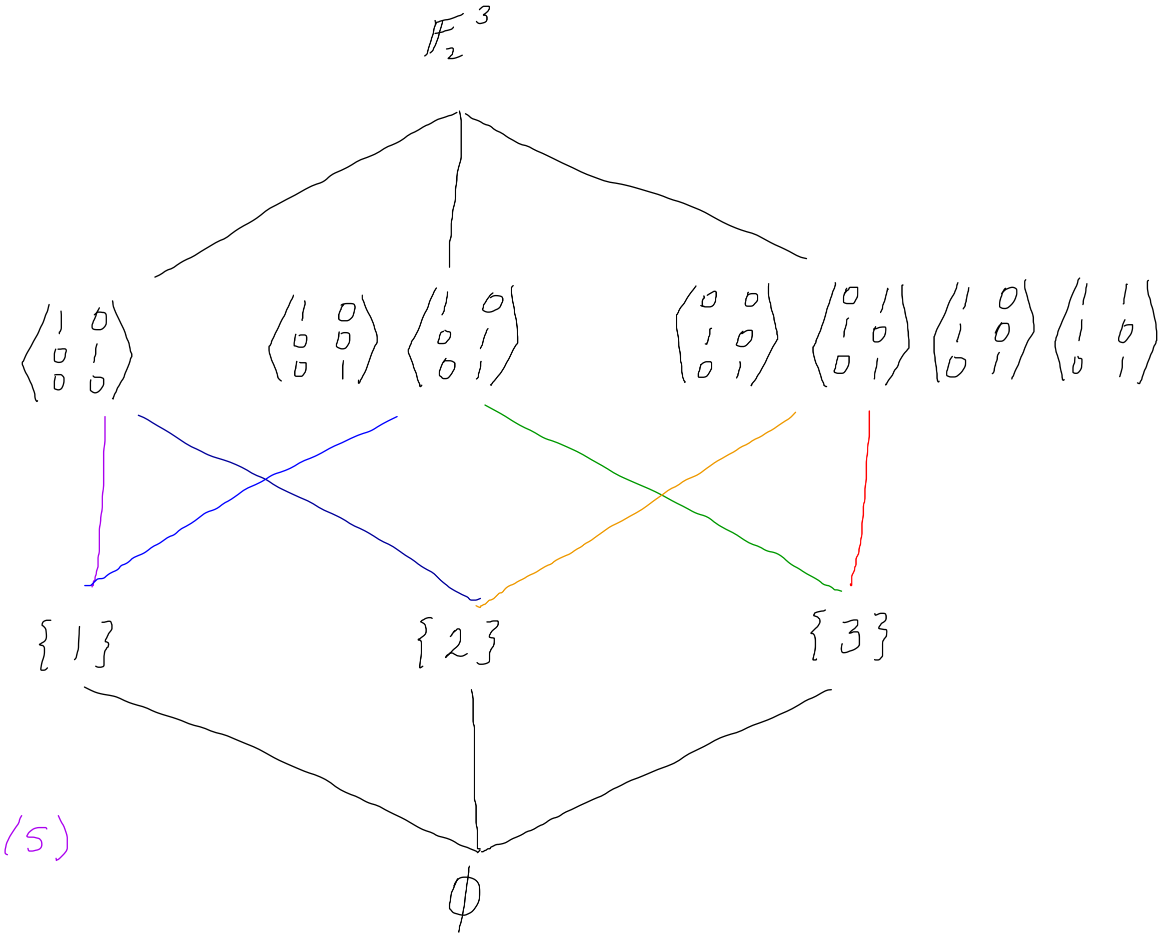


Part 4. The flag variety: The lattice *P(V)* Not baby case:

The field with two elements

$G = GL_3(\mathbb{F}_2)$ $\qquad$ $\mathcal{B} = \left\{ \begin{pmatrix} a_1 & c_1 & c_2 \\ 0 & a_2 & c_3 \\ 0 & 0 & a_3 \end{pmatrix} \,\middle|\, \begin{array}{l} c_1, c_2, c_3 \in \mathbb{F}_2 \\ a_1, a_2, a_3 \in \mathbb{F}_2^\times \end{array} \right\}$

$G/\mathcal{B}$ is maximal chains in the Fano plane



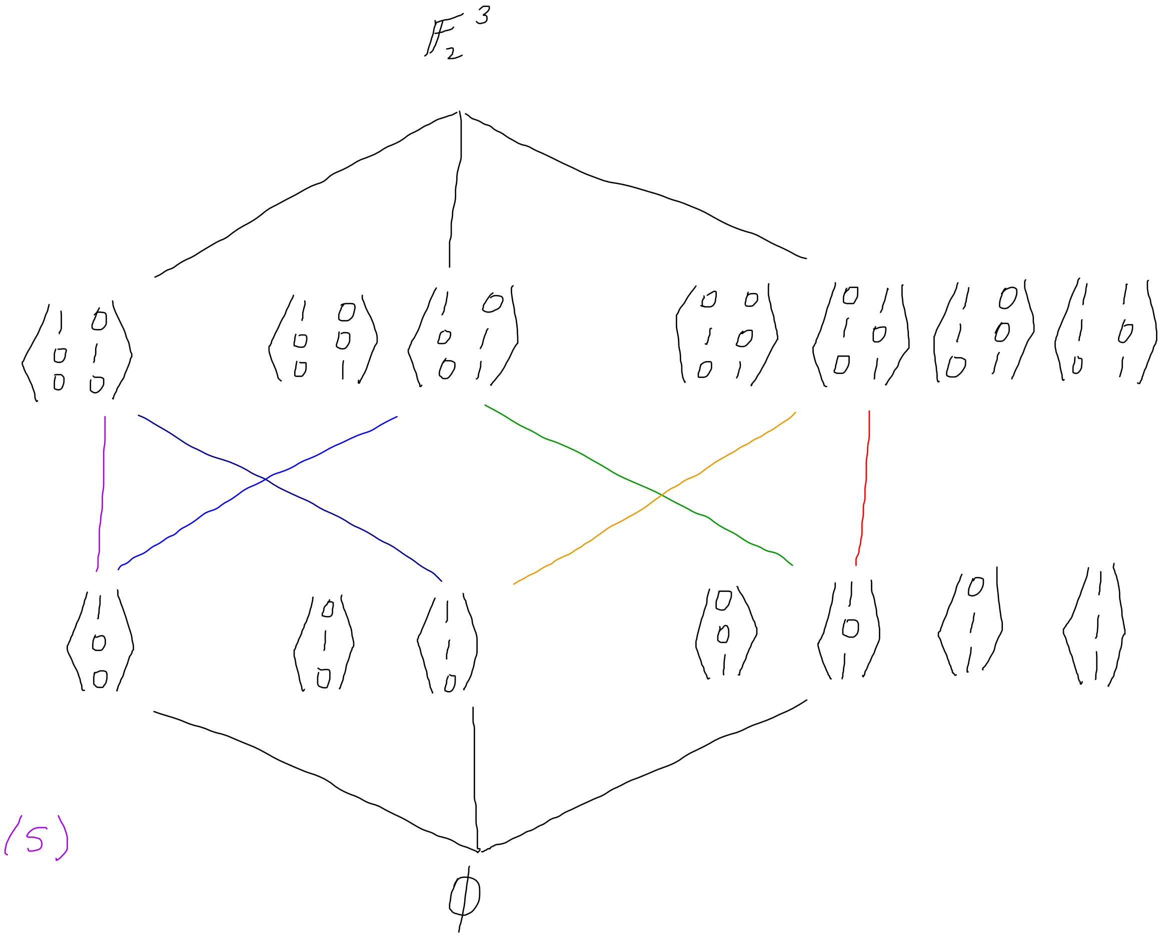
Let $\langle S \rangle = \mathrm{span}(S)$

Part 4. The flag variety: The lattice *P(V)* Not baby case:

The field with two elements

$G = GL_3(\mathbb{F}_2)$ $\quad$ $B = \left\{ \begin{pmatrix} a_1 & c_1 & c_2 \\ 0 & a_2 & c_3 \\ 0 & 0 & a_3 \end{pmatrix} \,\middle|\, \begin{array}{l} c_1, c_2, c_3 \in \mathbb{F}_2 \\ a_1, a_2, a_3 \in \mathbb{F}_2^\times \end{array} \right\}$

G/B is maximal chains in the Fano plane



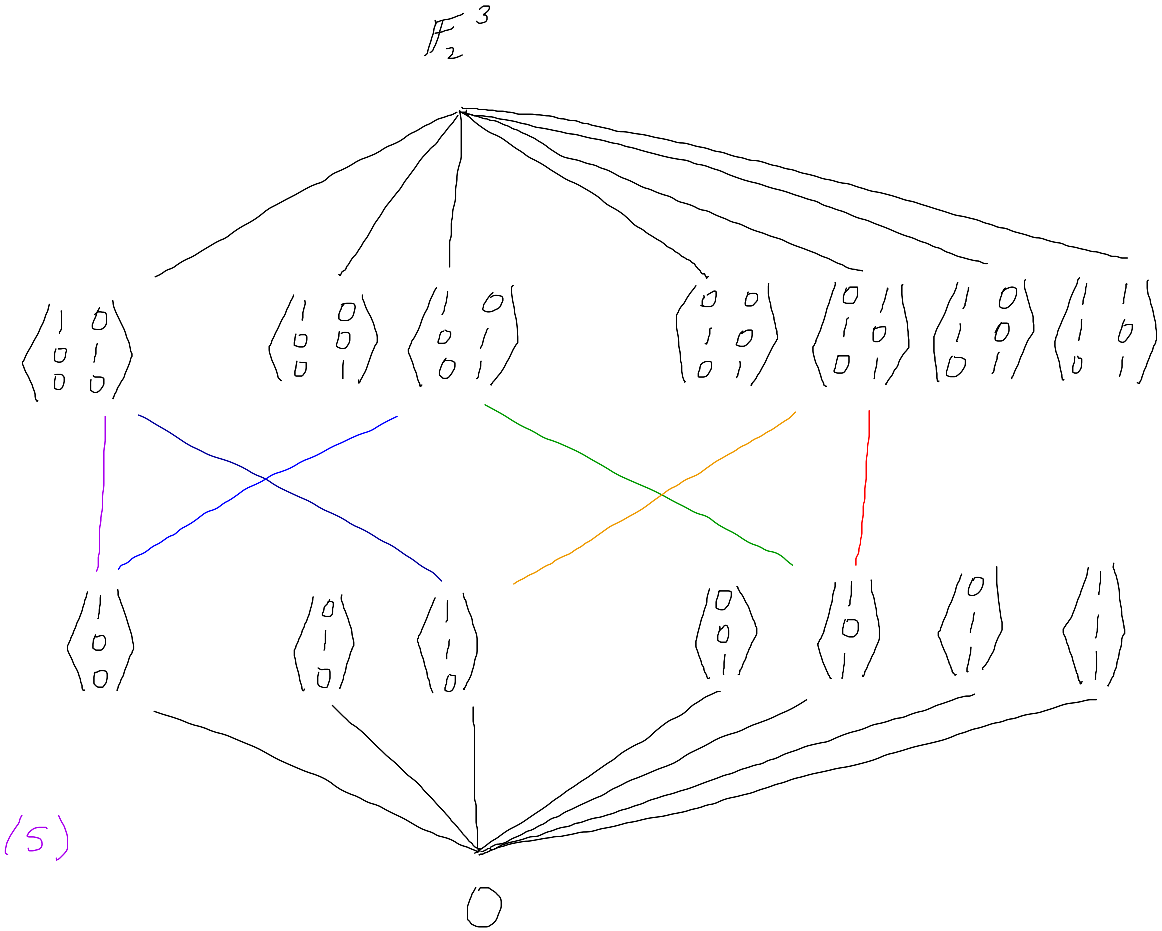
Let $\langle S \rangle = \mathrm{span}(S)$

Part 4. The flag variety: The lattice *P(V)* Not baby case:

The field with two elements

$G = GL_3(\mathbb{F}_2)$ $\qquad$ $\mathcal{B} = \left\{ \begin{pmatrix} a_1 & c_1 & c_2 \\ 0 & a_2 & c_3 \\ 0 & 0 & a_3 \end{pmatrix} \;\middle|\; \begin{matrix} c_1, c_2, c_3 \in \mathbb{F}_2 \\ a_1, a_2, a_3 \in \mathbb{F}_2^\times \end{matrix} \right\}$

$G/\mathcal{B}$ is maximal chains in the Fano plane



Let $\langle S \rangle = \mathrm{span}(S)$

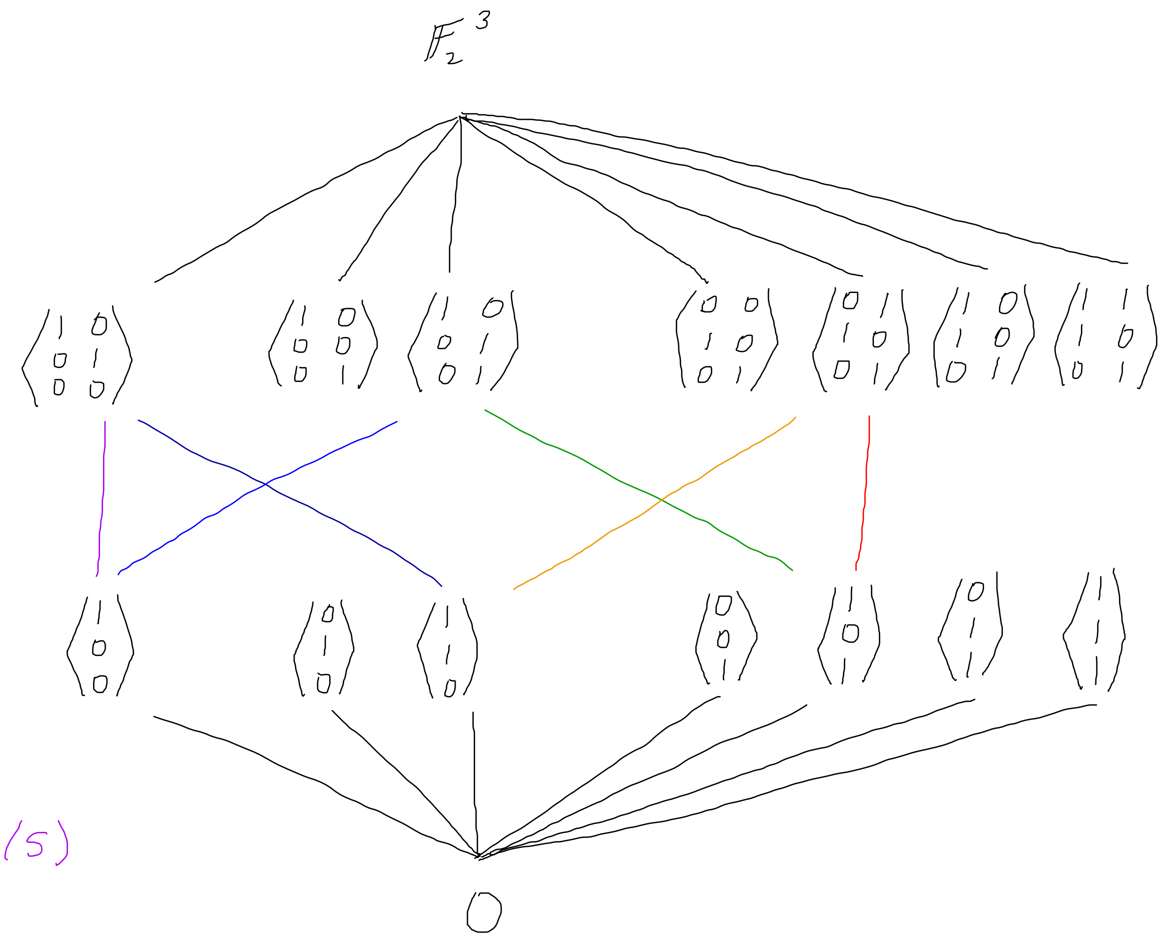
Part 4. The flag variety: The lattice *P(V)* Not baby case:

The field with two elements

$G = GL_3(\mathbb{F}_2)$

$$\mathcal{B} = \left\{ \begin{pmatrix} a_1 & c_1 & c_2 \\ 0 & a_2 & c_3 \\ 0 & 0 & a_3 \end{pmatrix} \;\middle|\; \begin{matrix} c_1, c_2, c_3 \in \mathbb{F}_2 \\ a_1, a_2, a_3 \in \mathbb{F}_2^\times \end{matrix} \right\}$$

$G/\mathcal{B}$ is maximal chains in the Fano plane



Let $\langle S \rangle = \mathrm{span}(S)$

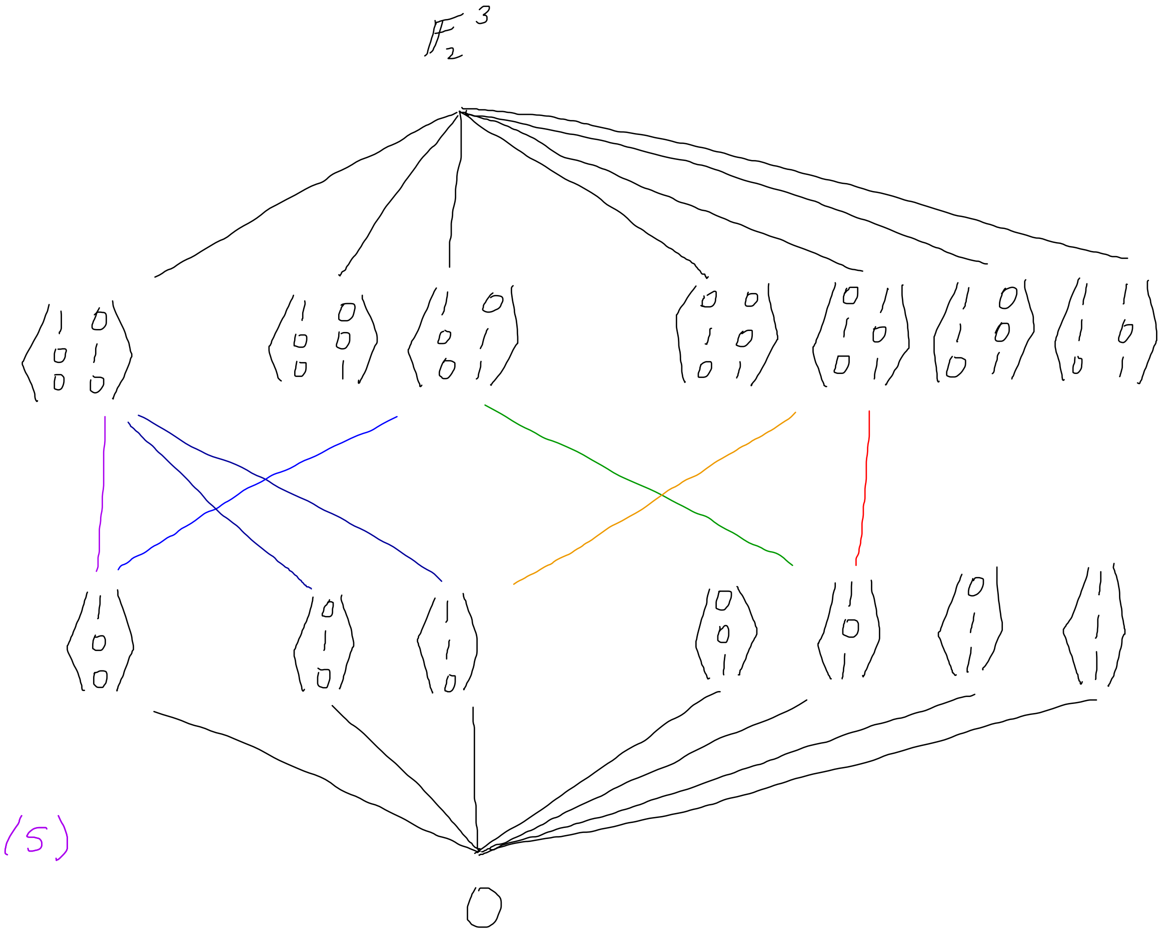
Part 4. The flag variety: The lattice *P(V)* Not baby case:

The field with two elements

$G = GL_3(\mathbb{F}_2)$

$$B = \left\{ \begin{pmatrix} a_1 & c_1 & c_2 \\ 0 & a_2 & c_3 \\ 0 & 0 & a_3 \end{pmatrix} \;\middle|\; \begin{matrix} c_1, c_2, c_3 \in \mathbb{F}_2 \\ a_1, a_2, a_3 \in \mathbb{F}_2^\times \end{matrix} \right\}$$

G/B is maximal chains in the Fano plane



Let $\langle S \rangle = \mathrm{span}(S)$

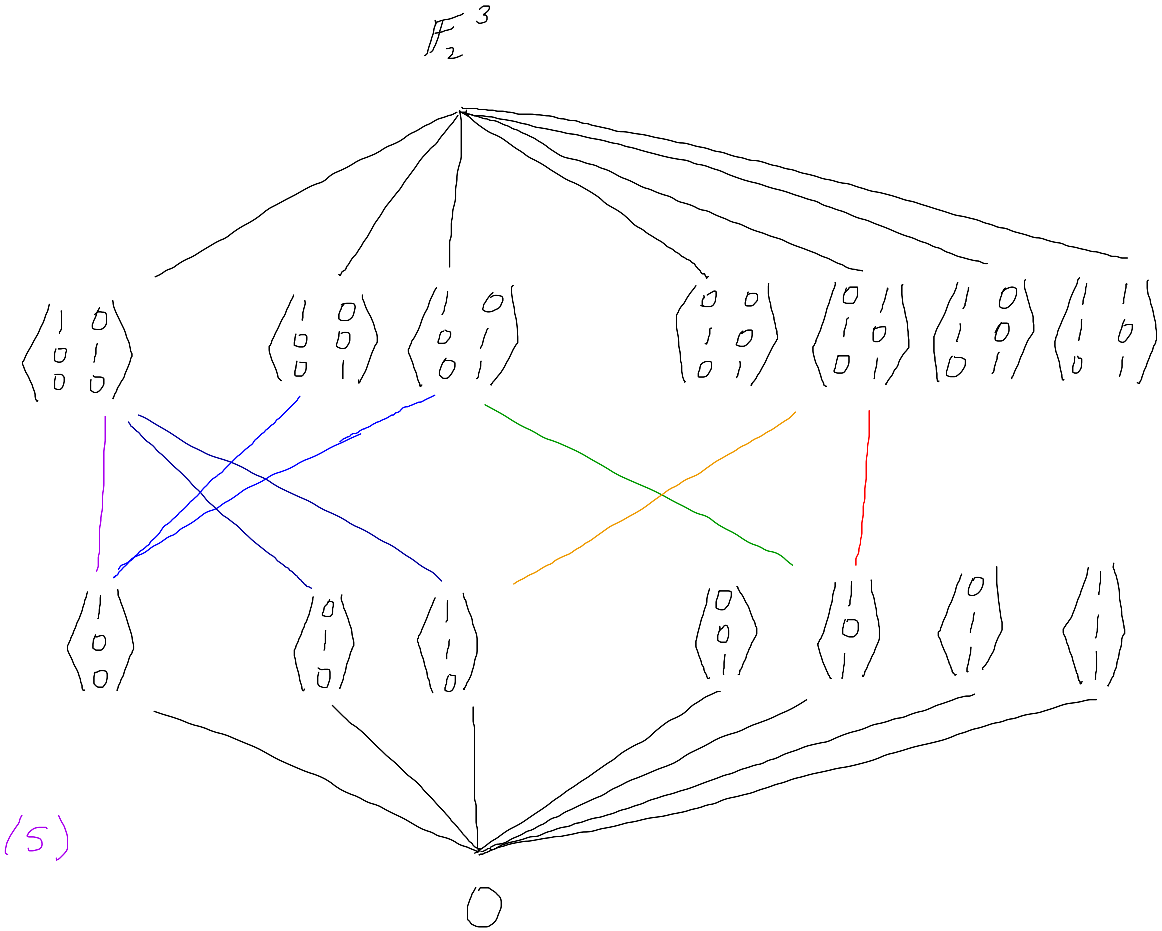
Part 4. The flag variety: The lattice *P(V)* Not baby case:

The field with two elements

$G = GL_3(\mathbb{F}_2)$

$$\mathcal{B} = \left\{ \begin{pmatrix} a_1 & c_1 & c_2 \\ 0 & a_2 & c_3 \\ 0 & 0 & a_3 \end{pmatrix} \;\middle|\; \begin{matrix} c_1, c_2, c_3 \in \mathbb{F}_2 \\ a_1, a_2, a_3 \in \mathbb{F}_2^{\times} \end{matrix} \right\}$$

$G/\mathcal{B}$ is maximal chains in the Fano plane



Let $\langle S \rangle = \mathrm{span}(S)$

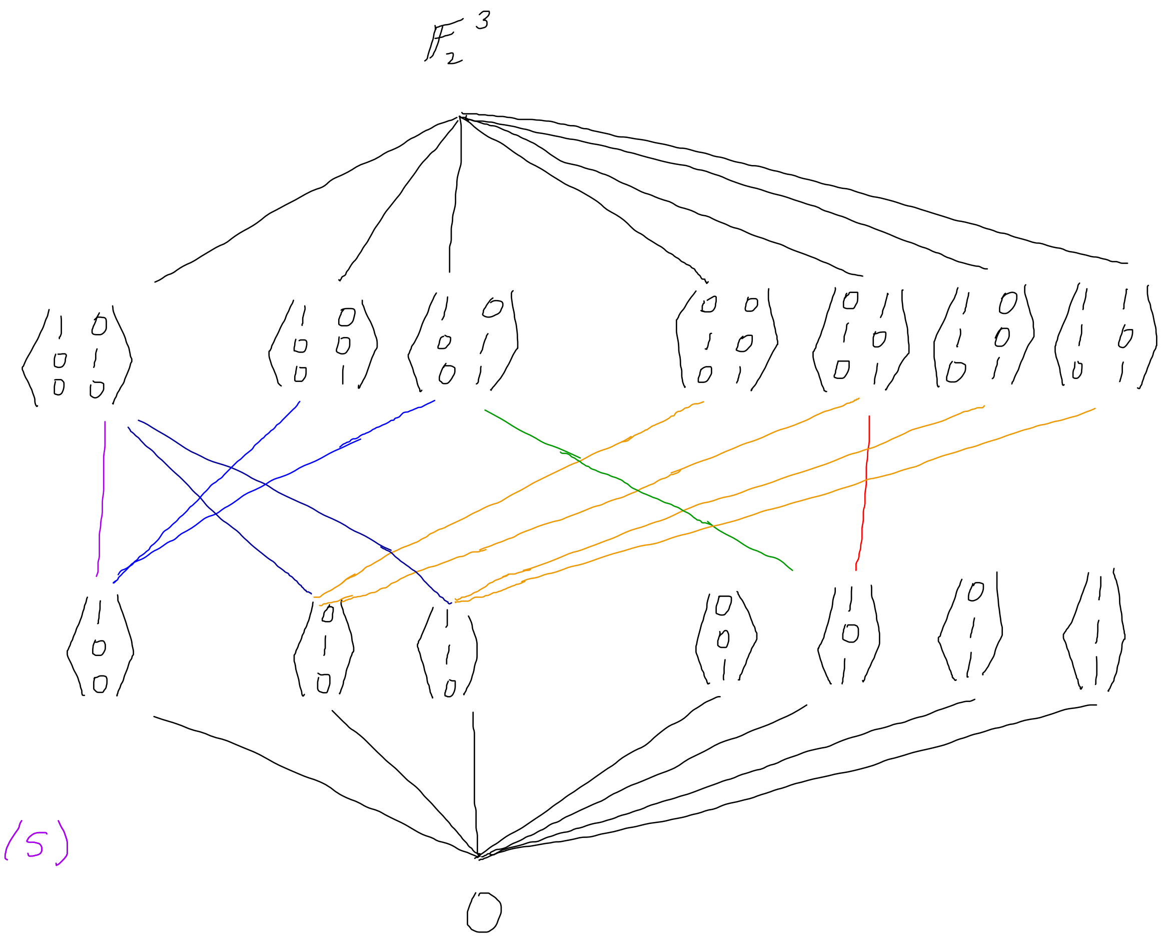
Part 4. The flag variety: The lattice *P(V)* Not baby case:

The field with two elements

$G = GL_3(\mathbb{F}_2)$

$$B = \left\{ \begin{pmatrix} a_1 & c_1 & c_2 \\ 0 & a_2 & c_3 \\ 0 & 0 & a_3 \end{pmatrix} \;\middle|\; \begin{array}{l} c_1, c_2, c_3 \in \mathbb{F}_2 \\ a_1, a_2, a_3 \in \mathbb{F}_2^{\times} \end{array} \right\}$$

G/B is maximal chains in the Fano plane



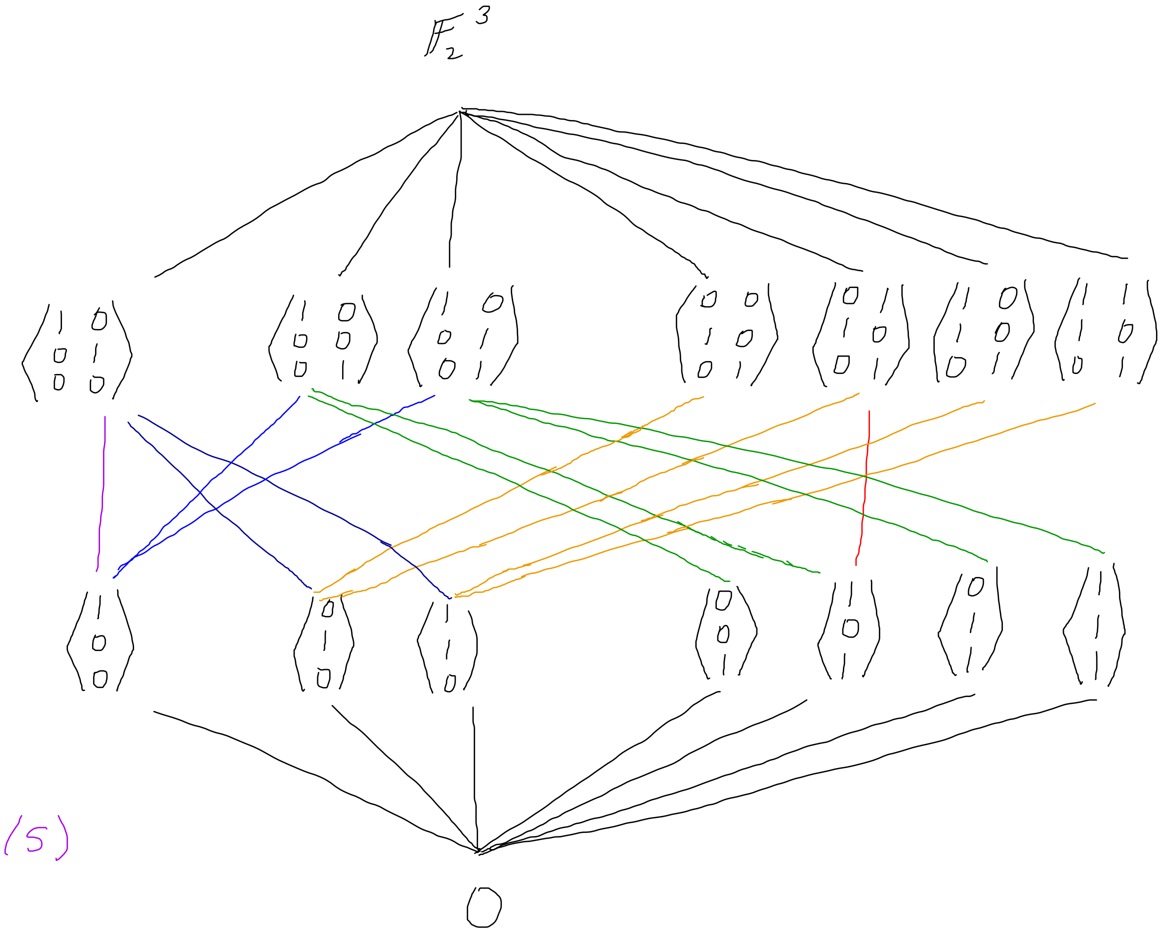
Let $\langle S \rangle = \mathrm{span}(S)$

Part 4. The flag variety: The lattice *P(V)* Not baby case:

The field with two elements

$G = GL_3(\mathbb{F}_2)$ $\qquad B = \left\{ \begin{pmatrix} a_1 & c_1 & c_2 \\ 0 & a_2 & c_3 \\ 0 & 0 & a_3 \end{pmatrix} \;\middle|\; \begin{matrix} c_1, c_2, c_3 \in \mathbb{F}_2 \\ a_1, a_2, a_3 \in \mathbb{F}_2^\times \end{matrix} \right\}$

G/B is maximal chains in the Fano plane



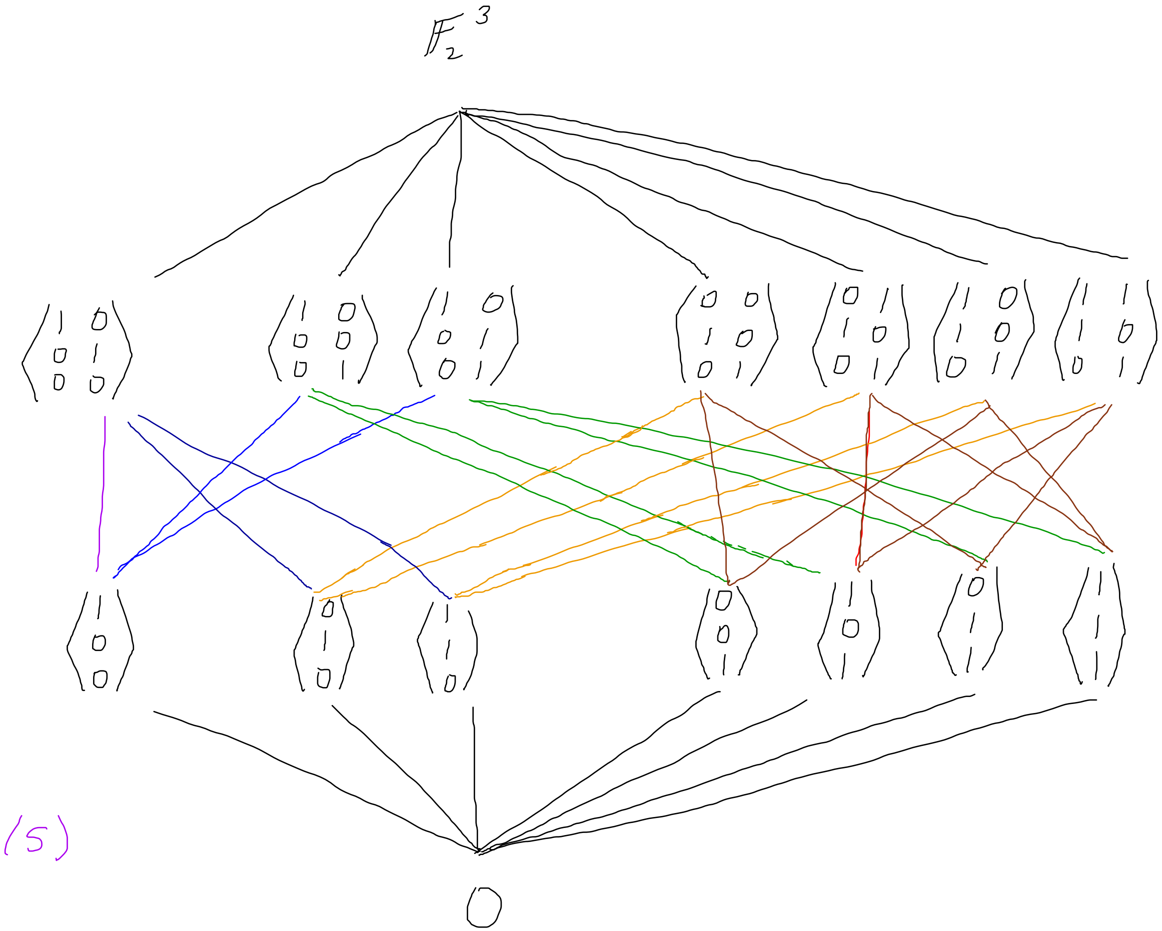
Let $\langle S \rangle = \text{span}(S)$

Part 4. The flag variety: The lattice *P(V)* Not baby case:

The field with two elements

$G = GL_3(\mathbb{F}_2)$ $\qquad$ $B = \left\{ \begin{pmatrix} a_1 & c_1 & c_2 \\ 0 & a_2 & c_3 \\ 0 & 0 & a_3 \end{pmatrix} \,\middle|\, \begin{matrix} c_1, c_2, c_3 \in \mathbb{F}_2 \\ a_1, a_2, a_3 \in \mathbb{F}_2^\times \end{matrix} \right\}$

G/B is maximal chains in the Fano plane



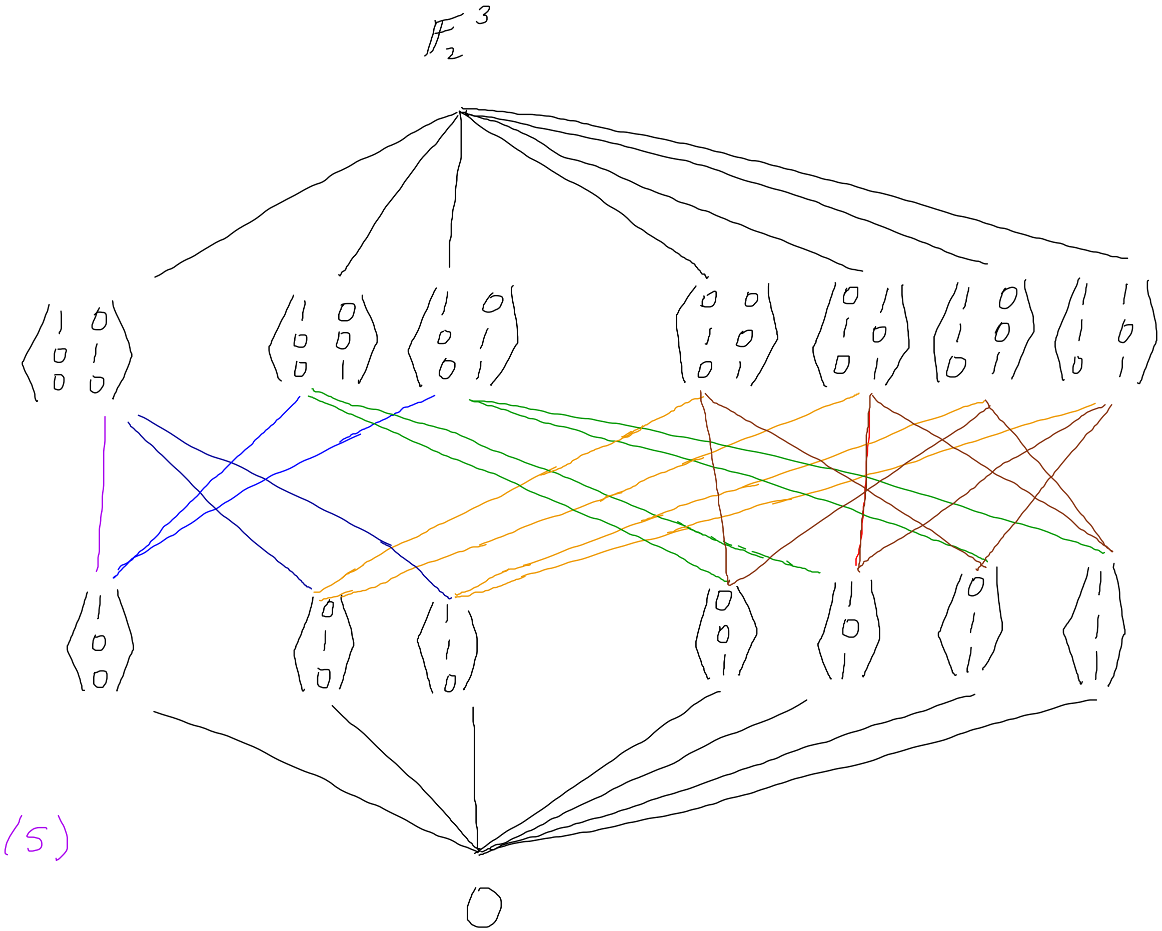
Let $\langle S \rangle = \mathrm{span}(S)$

Part 4. The flag variety: The lattice *P(V)* Not baby case:

The field with two elements

$G = GL_3(\mathbb{F}_2)$ $\qquad B = \left\{ \begin{pmatrix} a_1 & c_1 & c_2 \\ 0 & a_2 & c_3 \\ 0 & 0 & a_3 \end{pmatrix} \middle| \begin{array}{l} c_1, c_2, c_3 \in \mathbb{F}_2 \\ a_1, a_2, a_3 \in \mathbb{F}_2^\times \end{array} \right\}$

G/B is maximal chains in the Fano plane



points

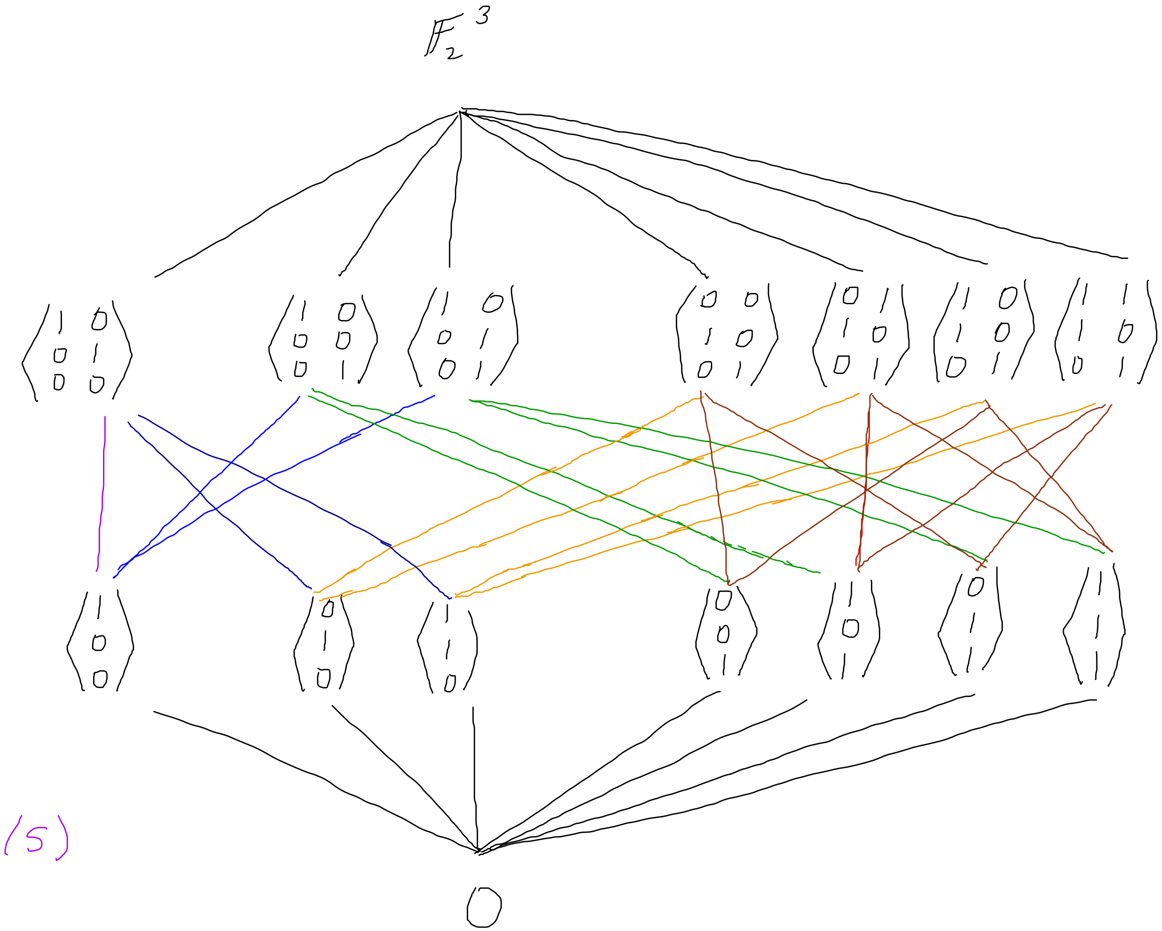
Let $\langle S \rangle = \mathrm{span}(S)$

Part 4. The flag variety: The lattice *P(V)* Not baby case:

The field with two elements

$G = GL_3(\mathbb{F}_2)$ $\qquad$ $\mathcal{B} = \left\{ \begin{pmatrix} a_1 & c_1 & c_2 \\ 0 & a_2 & c_3 \\ 0 & 0 & a_3 \end{pmatrix} \;\middle|\; \begin{array}{l} c_1, c_2, c_3 \in \mathbb{F}_2 \\ a_1, a_2, a_3 \in \mathbb{F}_2^\times \end{array} \right\}$

$G/\mathcal{B}$ is maximal chains in the Fano plane



lines

points

Let $\langle S \rangle = \mathrm{span}(S)$

Part 4. The flag variety: The lattice *P(V)* Not baby case:

The field with two elements

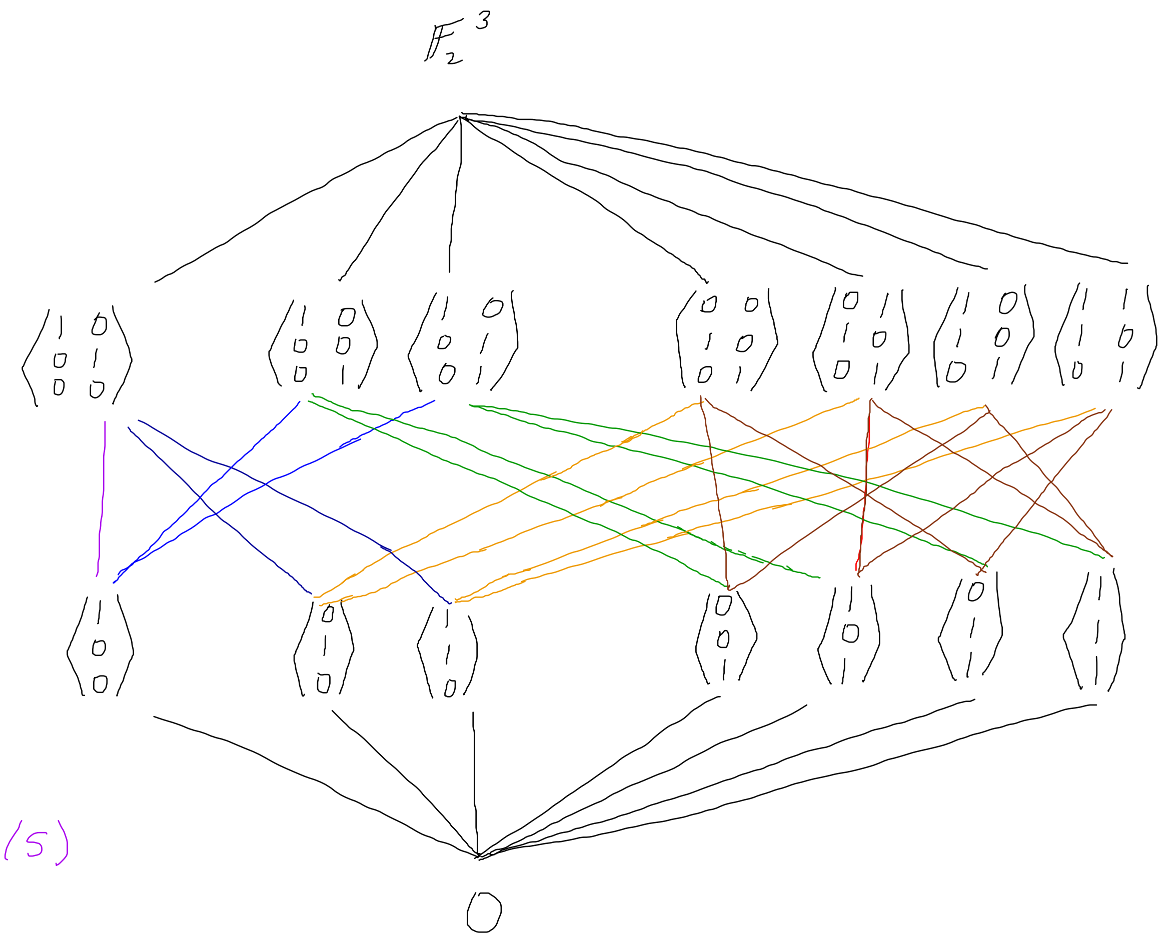
$G = GL_3(\mathbb{F}_2)$ $\qquad$ $B = \left\{ \begin{pmatrix} a_1 & c_1 & c_2 \\ 0 & a_2 & c_3 \\ 0 & 0 & a_3 \end{pmatrix} \middle| \begin{matrix} c_1, c_2, c_3 \in \mathbb{F}_2 \\ a_1, a_2, a_3 \in \mathbb{F}_2^\times \end{matrix} \right\}$

G/B is maximal chains in the Fano plane

plane

lines

points



Let $\langle S \rangle = \mathrm{span}(S)$

POINT AND LINE TO PLANE

BY WASSILY KANDINSKY

Wassily Kandinsky

POINT AND LINE TO PLANE

Sigma Series in Pure Mathematics
Volume 9

Donald E. Taylor

The Geometry of the Classical Groups

Heldermann Verlag

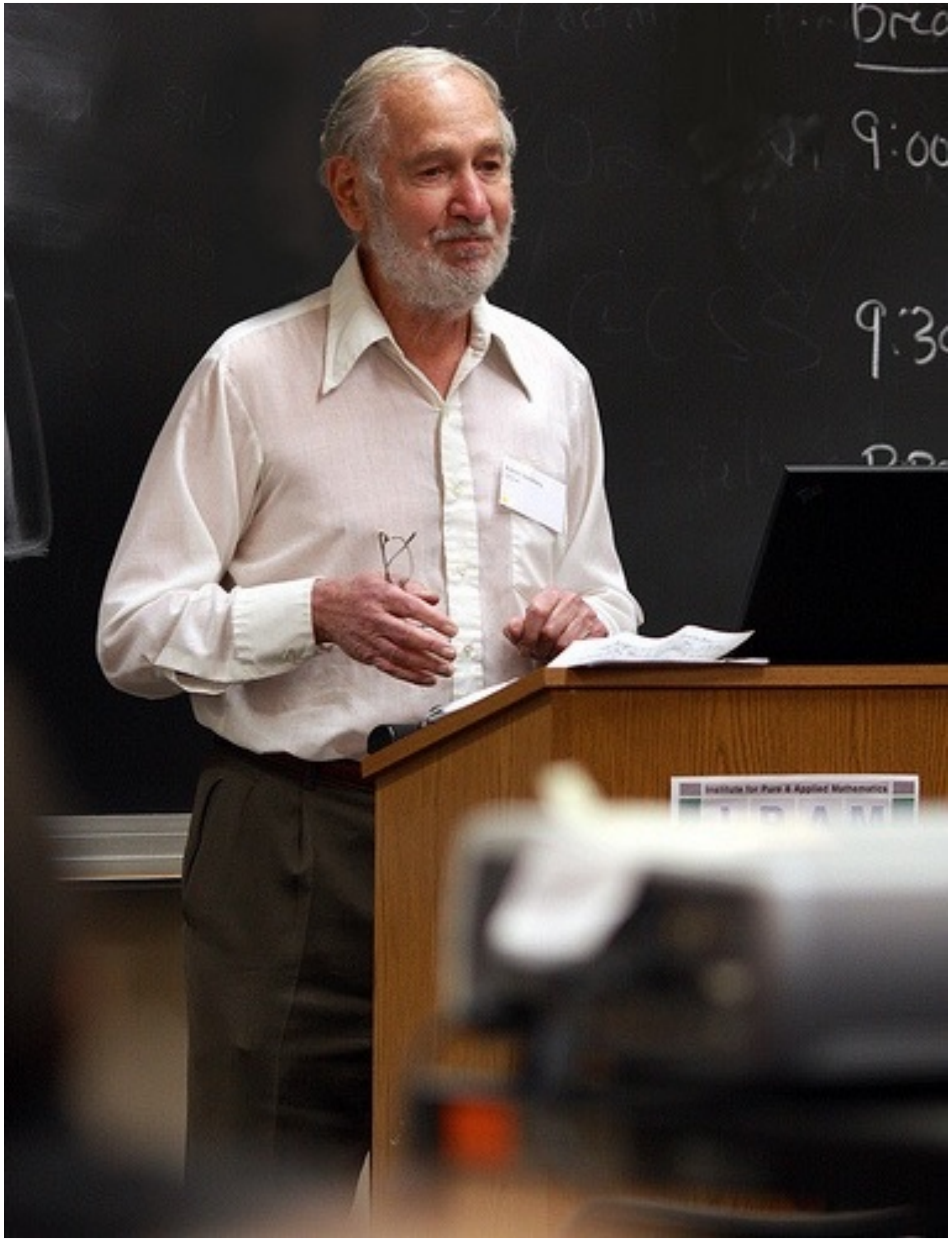
Lecture Notes in Mathematics

386

Jacques Tits

Buildings of Spherical Type and Finite BN-Pairs

Springer

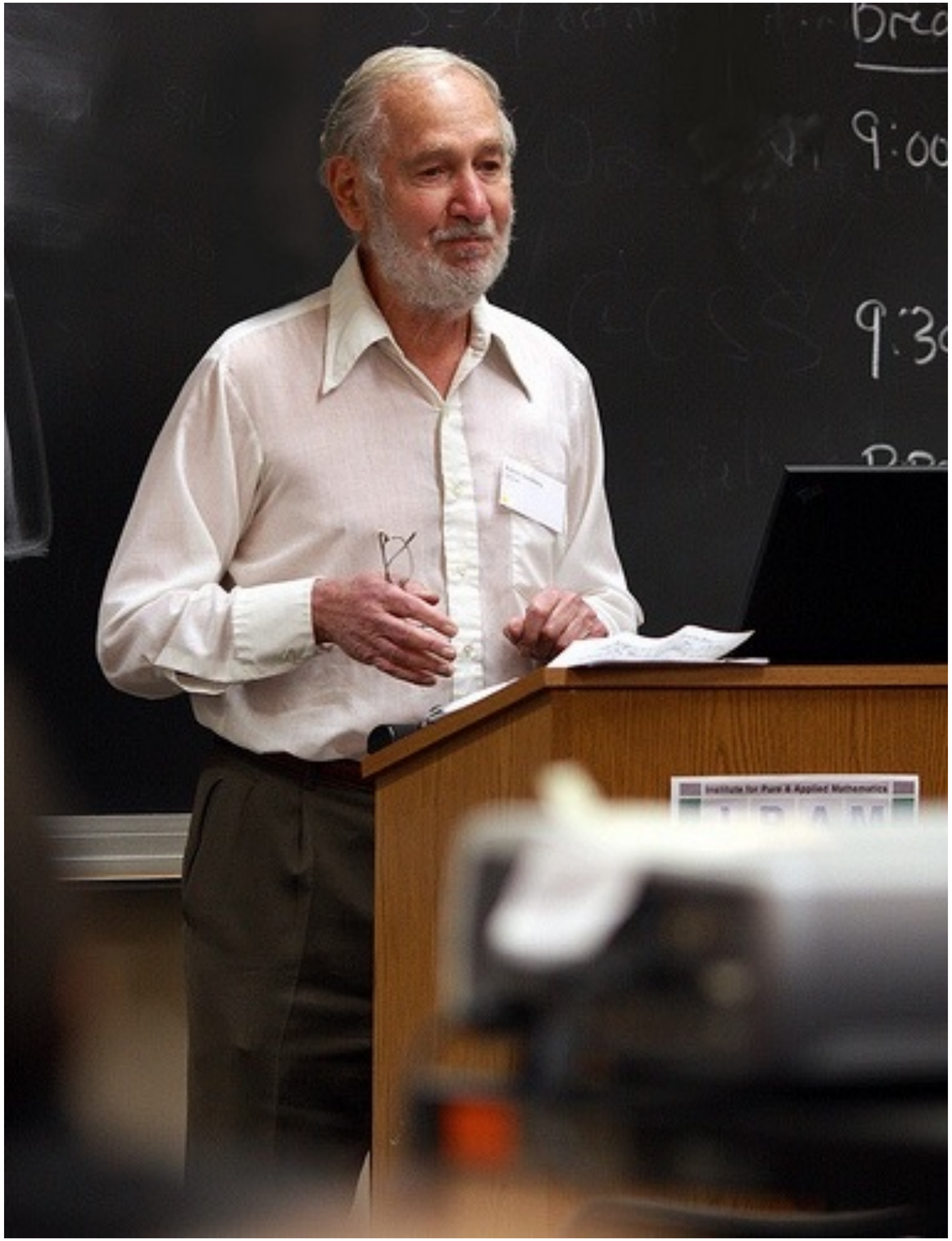


Robert Steinberg

Lectures on Chevalley groups

Yale University

1967



Robert Steinberg

1922–2014

in memory

Outline of this talk

Part 1. The flag variety

Part 2. The flag variety

Part 3. The flag variety

Part 4. The flag variety