Theta functions

Let
$$\mathbb{G}_1 = \{\tau \in \mathbb{C} \mid \operatorname{Im} \tau > 0\}.$$

Define the Jacobi Theta function
$$\Theta(z,q) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2} e^{2\pi i n z}, \quad \text{where } q = e^{i\pi\tau}$$
(see [AAR, (10.7.4)].)

Fix $\tau \in \mathbb{G}_1$ and let

$$\theta_4(z) = \Theta(z,q) = (q^2;q^2)_\infty (qe^{2iz};q^2)_\infty (qe^{-2iz};q^2)_\infty$$

(Jacobi's triple product formula [AAR (10.7.7)])

$$\theta_1(z) = \frac{1}{i} q^{\frac{1}{4}} e^{i\pi z}\, \Theta\!\left(z+\frac{\tau}{2}, q\right)$$

$$\theta_2(z) = q^{\frac{1}{4}} e^{i\pi z}\, \Theta\!\left(z+\frac{1}{2}+\frac{\tau}{2}, q\right)$$

$$\theta_3(z) = \Theta\!\left(z+\frac{1}{2}, q\right).$$

Let
$$k = \frac{\theta_2(0)^2}{\theta_3(0)^2} \quad \text{and} \quad k' = \frac{\theta_4(0)^2}{\theta_3(0)^2}$$

and
$$\operatorname{sn}(u) = \frac{\theta_3(0)}{\theta_2(0)} \frac{\theta_1\!\left(\frac{u}{\pi\theta_3(0)^2}\right)}{\theta_4\!\left(\frac{u}{\pi\theta_3(0)^2}\right)} \qquad \operatorname{cn}(u) = \sqrt{\frac{k'}{k}}\, \frac{\theta_2(v)}{\theta_4(v)}$$

$$\operatorname{dn}(u) = \sqrt{k'}\, \frac{\theta_3(v)}{\theta_4(v)} \quad \text{with } v = \frac{u}{\pi\theta_3(0)^2}$$

Relation between $\wp(z, \omega_1, \omega_2)$ and $\Theta(z)$.

~~Let~~ This is in Whittaker-Watson §21.73.

Let e_1, e_2, e_3 be the roots of $4y^3 - g_2 y - g_3 = 0$.
Let τ be determined by $\dfrac{e_1 - e_2}{e_1 - e_3} = \dfrac{\theta_4(0,\tau)^4}{\theta_3(0,\tau)^4}$.
Let A be given by

$$A^2 \theta(0)^4 = e_1 - e_2 .$$

Then

$$\wp(z \mid \omega_1, \omega_2) = A^2 \frac{\theta_2(Az,\tau)^2}{\theta_1(Az,\tau)^2} \theta_3(0,\tau)^2 \theta_4(0,\tau)^2 + e_1 .$$

Relation between $\wp(z, \omega_1, \omega_2)$ and $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$.

This is in Whittaker-Watson §22.351.

$$\wp(u; g_2, g_3) = e_3 + (e_1 - e_3) \frac{1}{\operatorname{sn}(u\sqrt{e_1 - e_3})^2}$$

where

$$k^2 = \frac{e_2 - e_3}{e_1 - e_3}$$

Addition formulas for theta functions

$$\theta_1(u)\theta_1(v)\theta_1(w)\theta_1(u+v+w)+\theta_4(u)\theta_4(v)\theta_4(w)\theta_4(u+v+w)$$

$$=\theta_4(0)\theta_4(u+v)\theta_4(u+w)\theta_4(v+w)$$

$$\theta_1(u)\theta_1(v)\theta_1(w)\theta_4(u+v+w)+\theta_4(u)\theta_4(v)\theta_1(w)\theta_1(u+v+w)$$

$$=\theta_4(0)\theta_4(u+v)\theta_1(u+w)\theta_1(v+w).$$

$$\theta_4(u-v)\theta_1(u+v)-\theta_4(u+v)\theta_1(u-v)=\frac{2\theta_1(v)\theta_2(u)\theta_3(u)\theta_4(v)}{\theta_2(0)\theta_3(0)}$$

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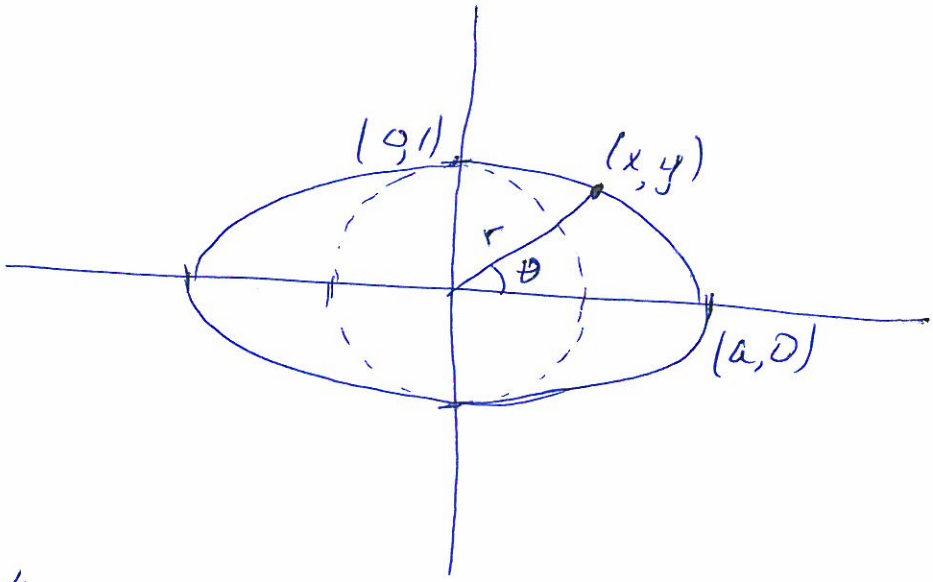
Elliptic Functions sn, cn, dn as Trigonometry



$u = \int_0^t r\, d\theta$, $\quad k = \sqrt{1 - \frac{1}{a^2}} \qquad k' = \sqrt{\frac{1}{a^2}}$

$sn(u) = y, \qquad cn(u) = \frac{x}{a}, \qquad dn(u) = \frac{r}{a}$

The ordinary trig functions are obtained in the limit $k \to 0$, i.e. $a \to 1$.
Assume $a \in \mathbb{R}_{\geq 1}$ so that $k \in [0,1]$.

Fact 1 $\quad cn^2 u + sn^2 u = 1$

Proof $\quad \left(\frac{x}{a}\right)^2 + y^2 = 1$, is the equation of the ellipse. //

Fact 2 $\quad dn^2 u + k^2 sn^2 u = 1.$

Proof $\quad \left(\frac{r}{a}\right)^2 + k^2 y^2 = \frac{y^2 + x^2}{a^2} + k^2 y^2 = \frac{y^2}{a^2} + \frac{x^2}{a^2} + \left(1 - \frac{1}{a^2}\right) y^2$

$= \left(\frac{x}{a}\right)^2 + y^2 = 1.$ //

<u>Fact 3</u>: $\frac{d}{du}\,\mathrm{sn}\,u = \mathrm{cn}\,u\;\mathrm{dn}\,u$

<u>Proof</u> $d\theta = \frac{1}{r^2}(x\,dy - y\,dx)$,

since $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ so that $d\theta = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}dy - \frac{y}{x^2}dx\right)$.

By definition of u,

$$du = r\,d\theta = \frac{1}{r}(x\,dy - y\,dx)$$

Since $\left(\frac{x}{a}\right)^2 + y^2 = 1$ then $\frac{x}{a^2}dx + y\,dy = 0$.

So $du = \frac{1}{r}\left(\frac{-x^2}{a^2 y} - y\right)dx$ and

$$du = \frac{1}{r}\left(x + \frac{a^2y^2}{x}\right)dy = \frac{a^2}{rx}dy$$

So $d(\mathrm{sn}) = dy = \frac{r}{a}\cdot\frac{a}{x}\,du = \mathrm{cn}(u)\,\mathrm{dn}(u)\,du$. and

$d(\mathrm{cn}) = \frac{1}{a}dx = \frac{1}{a}\,r\left(\frac{a^2y}{-x^2 - a^2y^2}\right)du = \frac{r}{a}\,\frac{y}{(-1)}\,du = -\mathrm{sn}(u)\,\mathrm{dn}(u)\,du$ //

<u>Fact 4</u>: $y = \mathrm{sn}\,u$ satisfies $\left(\frac{dy}{du}\right)^2 = (1-y^2)(1-k^2y^2)$.

<u>Proof</u> $\frac{d}{du}\mathrm{sn}\,u = \mathrm{cn}\,u\,\mathrm{dn}\,u = \sqrt{1-\mathrm{sn}^2u}\,\sqrt{1-k^2\mathrm{sn}\,u}$ //

<u>Fact 5</u> $u = c + \int \frac{dy}{\sqrt{1-y^2}\sqrt{1-k^2y^2}}$

<u>Proof</u> $du = \frac{dy}{\sqrt{1-y^2}\sqrt{1-k^2y^2}}$ //

<u>Fact 6</u>: $y = \operatorname{sn} u$ satisfies $y'' + (1+k^2)y - 2k^2y^3 = 0$.

<u>Proof</u> $\frac{d}{du}$ applied to $\left(\frac{dy}{du}\right)^2 = (1-y^2)(1-k^2y^2)$ is

$$2\frac{d^2y}{du^2}\frac{dy}{du} = (1-y^2)\left(-k^2 2y\frac{dy}{du}\right) - 2y\frac{dy}{du}(1-k^2y^2)$$

So

$$2y''y = -k^2 2yy' + 2k^2y^3y' - 2yy' + 2k^2y^3y'$$

$$= 2y'(-k^2y + k^2y^3 - y + k^2y^3)$$

So $y'' = -(1+k^2)y + 2k^2y^3$. //

Trig and elliptic trig identities

My 5 favorite trig identities are

(1) $e^{ix} = \cos x + i \sin x$

(2) $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$

(3) $\cos^2 x + \sin^2 x = 1$

(4) $\cos(x+y) = \cos x \cos y - \sin x \sin y$

$\sin(x+y) = \sin x \cos y + \cos x \sin y$

(5) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

The elliptic analogues are:

(2) $\mathrm{cn}(-u) = \mathrm{cn}\, u, \quad \mathrm{sn}(-u) = -\mathrm{sn}\, u, \quad \mathrm{dn}(-u) = \mathrm{dn}\, u$

$\mathrm{cn}(2K-u) = -\mathrm{cn}\, u, \quad \mathrm{sn}(2K-u) = \mathrm{sn}\, u, \quad \mathrm{dn}(2K-u) = \mathrm{dn}\, u$

$\mathrm{cn}(2iK'-u) = -\mathrm{cn}\, u, \quad \mathrm{sn}(2iK'-u) = -\mathrm{sn}\, u, \quad \mathrm{dn}(2iK'-u) = -\mathrm{dn}\, u$

where

$$K = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} \quad \text{and} \quad K' = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-(k')^2\sin^2\varphi}}$$

$$= \tfrac{1}{2}\pi\, \theta_3(0)^2 \qquad\qquad = -\frac{i\pi\tau}{2}\, \theta_3(0)^2$$

(3) $cn^2 u + sn^2 u = 1$ and $dn^2 u + k^2 sn^2 u = 1$.

(4) $$cn(u+v) = \frac{cn\,u\, cn\,v - sn\,u\, dn\,u\, sn\,v\, dn\,v}{1 - k^2 sn^2 u\, sn^2 v}$$

$$dn(u+v) = \frac{dn\,u\, dn\,v - k^2 sn\,u\, cn\,u\, sn\,v\, cn\,v}{1 - k^2 sn^2 u\, sn^2 v}$$

$$sn(u+v) = \frac{sn\,u\, cn\,v\, dn\,v + cn\,v\, dn\,u\, sn\,v}{1 - k^2 sn^2 u\, sn^2 v}.$$

Derivatives of the trig and elliptic trig functions

The favorite derivatives

$$\frac{d}{dx}\cos x = -\sin x \quad \text{and} \quad \frac{d}{dx}\sin x = \cos x$$

generalize to

$$\frac{d}{du}\, cn\,u = -sn\,u\, dn\,u,$$

$$\frac{d}{du}\, dn\,u = -k^2 sn\,u\, cn\,u$$

$$\frac{d}{du}\, sn\,u = cn\,u\, dn\,u.$$

Relations between hyperbolic functions and trig functions

$\sinh x = -i\sin(ix)$ and $\cosh x = \cos(ix)$

~~cosh x = cos(ix)~~

Elliptic Functions

Theorem

(a) $sn^2(u) + cn^2(u) = 1$

(b) $k^2 sn^2(u) + dn^2(u) = 1$

(c) $\frac{d}{du} sn(u) = cn(u)\, dn(u)$

$\frac{d}{du} cn(u) = -sn(u)\, dn(u)$

$\frac{d}{du} dn(u) = -k^2 sn(u)\, cn(u).$

(d) If $y = sn(u)$ then $\left(\frac{dy}{du}\right)^2 = (1-y^2)(1-k^2y^2).$

(e) $sn(-u) = -sn(u)$

$cn(-u) = cn(u)$

$dn(-u) = dn(u)$

(f) $sn(2K-u) = sn(u),$

$cn(2K-u) = -cn(u)$ where $K = \frac{1}{2}\pi\, \theta_3^2(0) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$

$dn(2K-u) = dn(u)$

(g) $sn(2iK'-u) = -sn(u),$

$cn(2iK'-u) = -cn(u),$ where $K' = \frac{i\pi\tau}{2}\theta_3^2(0) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k'^2\sin^2\varphi}}$

$dn(2iK'-u) = -dn(u),$

Definitions of sn(u), cn(u), dn(u)

$$\mathrm{sn}(u,k) = \frac{1}{\sqrt{k}} \frac{\theta_1\left(\frac{u}{2K_k}\right)}{\theta_4\left(\frac{u}{2K_k}\right)}$$

$$\mathrm{cn}(u,k) = \sqrt{\frac{k'}{k}} \frac{\theta_2\left(\frac{u}{2K_k}\right)}{\theta_4\left(\frac{u}{2K_k}\right)}$$

$$\mathrm{dn}(u,k) = \sqrt{k'} \frac{\theta_3\left(\frac{u}{2K_k}\right)}{\theta_4\left(\frac{u}{2K_k}\right)}$$

with

$$K_k = \tfrac{1}{2}\pi\,\theta_3^2(0) \quad \text{and} \quad K_k' = -\frac{i\pi\tau}{2}\,\theta_3^2(0)$$

$$k = \frac{\theta_2^2(0)}{\theta_3^2(0)} \quad \text{and} \quad k' = \frac{\theta_4^2(0)}{\theta_3^2(0)}$$

Here $\tau \in \mathbb{C}$, $\mathrm{Im}(\tau) > 0$ and $q = e^{i\pi\tau}$

$$\theta(z) = \theta(z,q) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2} e^{2\pi i n z}$$

and

$$\theta_4(z) = \theta(z),$$

$$\theta_3(z) = \theta\left(z+\tfrac{1}{2}\right)$$

$$\theta_2(z) = q^{\frac{1}{4}} e^{i\pi z}\,\theta\left(z+\tfrac{1}{2}+\tfrac{\tau}{2}\right)$$

$$\theta_1(z) = \frac{1}{i} q^{\frac{1}{4}} e^{i\pi z}\,\theta\left(z+\tfrac{\tau}{2}\right).$$

Elliptic functions

Theorem:

(a) $\mathrm{sn}^2 u + \mathrm{cn}^2 u = 1.$ [Sc(7)] [TF, I.15]

(b) $k^2 \mathrm{sn}^2 u + \mathrm{dn}^2 u = 1$ [Sc(8)] [TF, I.15]

(c) $\frac{d}{du} \mathrm{sn}\, u = \mathrm{cn}\, u \, \mathrm{dn}\, u$ [Sc(9)] [TF I.19]

(d) $\frac{d}{du} \mathrm{cn}\, u = -\mathrm{sn}\, u \, \mathrm{dn}\, u$ [Sc(10)] [TF I.19]

(e) $\frac{d}{du} \mathrm{dn}\, u = -k^2 \mathrm{sn}\, u \, \mathrm{cn}\, u$ [Sc(11)] [TF I.19]

(f) If $y = \mathrm{sn}\, u$ then $\left(\frac{dy}{du}\right)^2 = (1-y^2)(1-k^2y^2)$ [Sc(12)] [TF, I.13]

(g) $\mathrm{sn}(-u) = -\mathrm{sn}\, u$

$\mathrm{cn}(-u) = \mathrm{cn}\, u$

$\mathrm{dn}(-u) = \mathrm{dn}\, u$

(h) $\mathrm{sn}(2K-u) = \mathrm{sn}\, u,$

$\mathrm{cn}(2K-u) = -\mathrm{cn}\, u,$ where $K = \frac{1}{2}\pi\, \theta_3^2(0) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$

$\mathrm{dn}(2K-u) = \mathrm{dn}\, u,$

(i) $\mathrm{sn}(2iK'-u) = -\mathrm{sn}\, u,$

$\mathrm{cn}(2iK'-u) - \mathrm{cn}\, u,$ where $K' = \frac{-i\pi\tau}{2}\, \theta_3^2(0) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k'^2\sin^2\varphi}}$

$\mathrm{dn}(2iK'-u) = -\mathrm{dn}\, u,$

Theta functions

$\tau \in \mathbb{C}$, $\mathrm{Im}(\tau) > 0$ and

$$q = e^{i\pi\tau}.$$

Let

$$\theta(z) = \theta(z,q) = \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2} e^{2\pi i n z}$$

Then let

$$\theta_4(z) = \theta(z)$$

$$\theta_3(z) = \theta(z + \tfrac{1}{2})$$

$$\theta_2(z) = q^{\frac{1}{4}} e^{i\pi z} \theta(z + \tfrac{1}{2} + \tfrac{\tau}{2})$$

$$\theta_1(z) = \tfrac{1}{i} q^{\frac{1}{4}} e^{i\pi z} \theta(z + \tfrac{\tau}{2}).$$

and

$$k = \frac{\theta_2^2(0)}{\theta_3^2(0)} \quad \text{and} \quad k' = \frac{\theta_4^2(0)}{\theta_3^2(0)}$$

Theorem (a) $k^2 + k'^2 = 1$. [TF, I.9]

[TF, after I.12] (b) If $k^2 \neq 0, 1$ then τ is uniquely determined by k.

Theorem (a) $\theta(z+1) = \theta(z)$

[TF, I.2] (b) $\theta(z+\tau) = -\dfrac{e^{-2\pi i z}}{q}\theta(z)$

[TF, I.6] (c) $\theta_4(u-v)\theta_1(u+v) - \theta_4(u+v)\theta_1(u-v) = \dfrac{2\theta_1(v)\theta_2(u)\theta_3(u)\theta_4(v)}{\theta_2(0)\theta_3(0)}$

(d) [TF, I.4]

(e) [TF, I.5]

From Andrews, Askey, Roy. ③

$$\theta_1(z) = -i \sum_{n\in\mathbb{Z}} (-1)^n q^{\left(\frac{n+1}{2}\right)^2} e^{(2n+1)iz}$$

$$\theta_2(z) = \theta_1\left(z+\frac{\pi}{2}\right)$$

$$\theta_3(z) = \sum_{n\in\mathbb{Z}} q^{n^2} e^{2inz}$$ and $q = e^{i\pi\tau}$ with $\mathrm{Im}(\tau) > 0$ so that $|q| < 1$.

$$\theta_4(z) = \theta_3\left(z+\frac{\pi}{2}\right) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2} e^{2inz}$$

They give product formulas:

[AAR, (10.7.7)] $\theta_4(z) = (q^2;q^2)_\infty (qe^{2iz};q^2)_\infty (qe^{-2iz};q^2)_\infty$

equivalent to the Jacobi triple product formula.

The rest of ~~their~~ treatment seems not so useful.

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