# Combinatorics, Representations, Homogeneous spaces and Elliptic cohomology

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University of Melbourne

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## People in my corridor at Melbourne

Hyam Rubinstein

Martina Lanini

Lawrence Reeves

Craig Hodgson

Paul Norbury

Nora Ganter

Craig Westerland

Paul Sobaje

Jan de Gier

Omar Ortiz

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## People thinking about this subject with me

Harsh Pittie

Stephen Griffeth

Nora Ganter

Craig Westerland

Omar Ortiz

Katie Bowles

Nicolas Thiery

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Google: Pure position Melbourne

# Cohomology of the flag variety

$$H_T(G/B)$$

generalize

generalize

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$$H_T(G/B)$$

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- K-theory $K_T$
- Elliptic cohomology $Ell_T$
- Cobordism $\Omega_T$

- reductive algebraic groups
- compact Lie groups
- p-compact groups

# Schubert Calculus: Cohomology of the flag variety

$$[X_w] \in H_T(G/B)$$

## Linear algebra Theorem!

$$G = GL_n(\mathbb{C}) \supseteq B = \left\{ \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \right\}$$

$$W_0 = S_n = \{n \times n \text{ permutation matrices}\}$$

$$G = \bigsqcup_{w \in W_0} BwB$$

$X_w = \overline{BwB}$ are the Schubert Varieties

# Schubert Varieties $X_w$

Special cases are:

- Projective space $\mathbb{P}^n$
- Grassmannians $Gr_k(n)$
- Classical flag varieties $Fl(n)$

Most $X_w$ are singular,

but not too badly singular.

$$Fl(n) = \{(0 \subseteq V_1 \subseteq \cdots \subseteq V_n) \mid \dim_{\mathbb{C}} V_i = i\}$$

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$$H_T(G/B)$$

**Borel model:** $H_T(G/B) = S \otimes_{S^{W_0}} S$

**GKM model:** $H_T(G/B) = (S \otimes S) \cdot 1$

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Solution:

Change S

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# Borel presentation $H_T(G/B) = S \otimes_{S^{W_0}} S$

$$\left\{ \begin{array}{c} \mathbb{Z}\text{-reflection groups} \\ (W_0, \mathfrak{h}^*_{\mathbb{Z}}) \end{array} \right\}$$

equivalences of categories

$$\left\{ \begin{array}{l} G \text{ reductive algebraic group} \\ \cup\!| \\ B \quad \text{Borel subgroup} \\ \cup\!| \\ T \quad \text{maximal torus} \end{array} \right\}$$

$$S = S(\mathfrak{h}^*_{\mathbb{Z}}) = H_T(pt) = H(BT)$$

$$S^{W_0} = H_G(pt) = H(BG)$$

Borel presentation $H_T(G/B) = S \otimes_{S^{W_0}} S$

$$\left\{\begin{array}{l} U \text{ compact Lie group} \\ \cup\!\shortmid \\ D \text{ maximal torus} \end{array}\right\} \longleftrightarrow \left\{\begin{array}{c} \mathbb{Z}\text{-reflection groups} \\ (W_0, \mathfrak{h}^*_{\mathbb{Z}}) \end{array}\right\}$$

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$$S = S(\mathfrak{h}^*_{\mathbb{Z}}) = \mathbb{C}[x_1, \dots, x_n]$$

$$W_0 = N(T)/T \qquad \mathfrak{h}^*_{\mathbb{Z}} = \mathrm{Hom}(T, \mathbb{C}^\times)$$

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**Fibration sequence**

$$\begin{tikzcd} U/D \arrow[r] \arrow[d, equal] & BD \arrow[r] \arrow[d, equal] & BU \arrow[d, equal] \\ G/B \arrow[r] & BT \arrow[r] & BG \end{tikzcd}$$

# Cohomology of the flag variety

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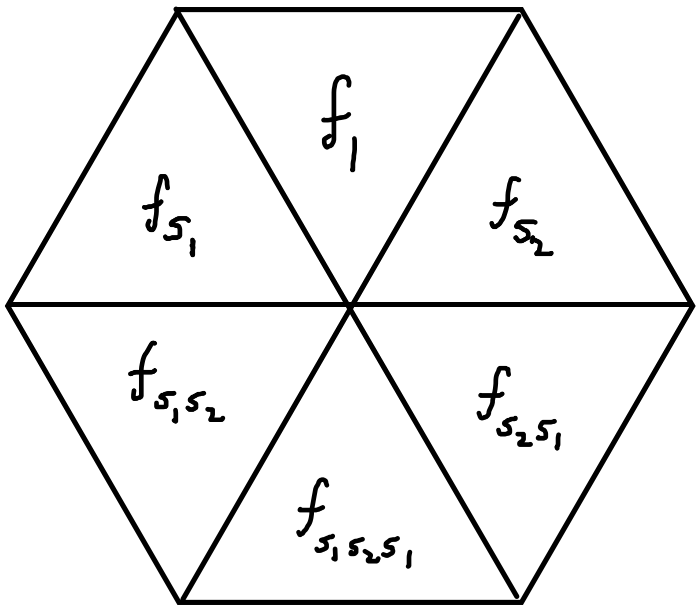
## GKM model

$$H_T(G/B) = (S\otimes S)\cdot 1$$

Put an element $f_w \in S$ in each chamber.

- addition and multiplication are pointwise
- $S\otimes S$ acts on $\bigoplus_{w\in W_0} S$



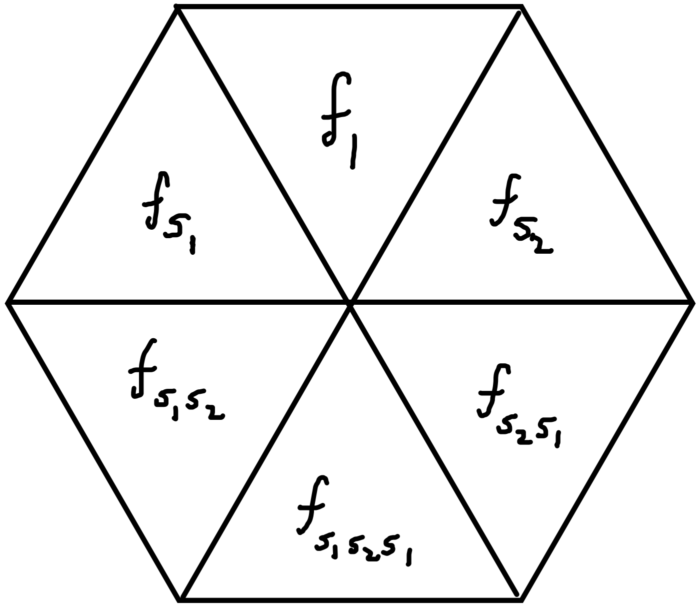

$$\in \bigoplus_{w\in W_0} S$$

# GKM model

T-fixed points: $\imath_w : pt \longrightarrow G/B$

$* \longmapsto wB$

$$\imath^* : H_T(G/B) \xrightarrow{\bigoplus_{w\in W_0} \imath_w^*} \bigoplus_{w\in W_0} H_T(pt)$$

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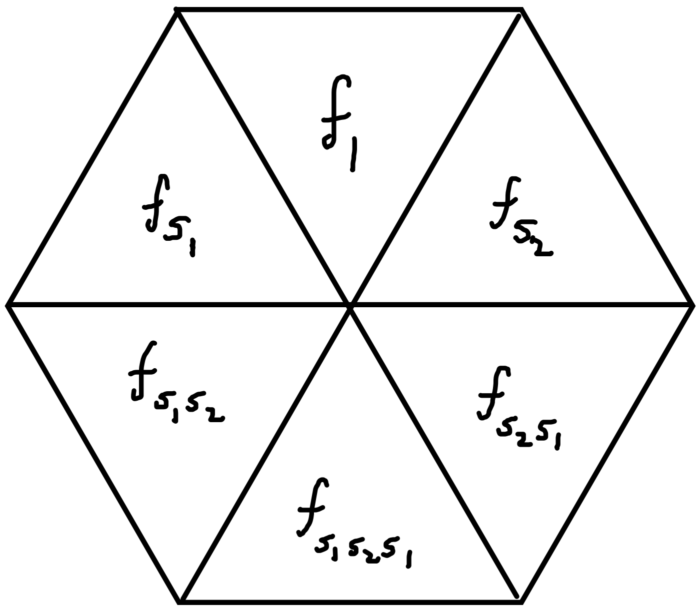
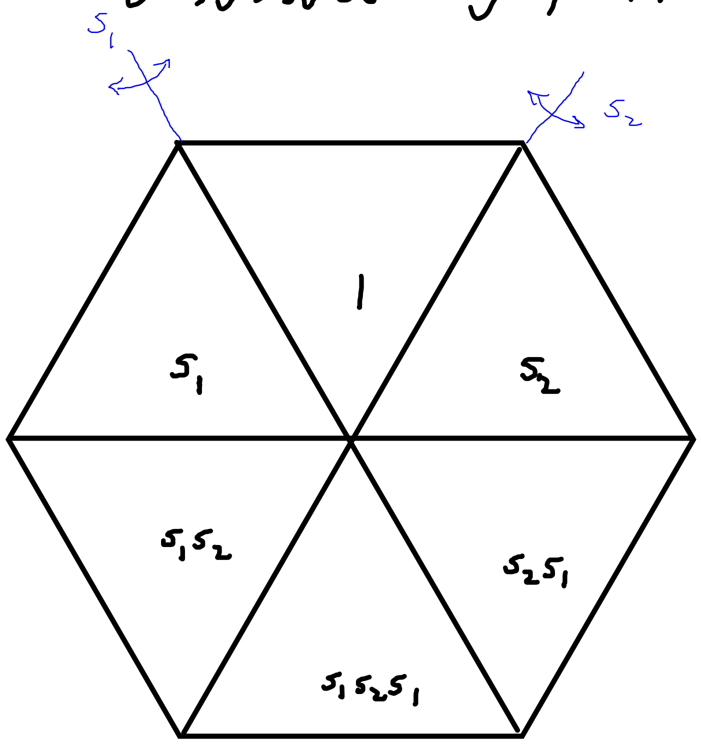



$$\in \bigoplus_{w \in W_0} S$$

$$\underline{H_T(G/B) = (S \otimes S)\cdot 1}$$ For $G = GL_3(\mathbb{C})$

$W_0 = \langle s_1, s_2 \mid s_i^2 = 1,\ s_1 s_2 s_1 = s_2 s_1 s_2 \rangle$ acts on

$H_T(pt) = S = \mathbb{C}[y_1, y_2, y_3]$ by permuting $y_1, y_2, y_3$



Put a polynomial $f_w \in \mathbb{C}[y_1, y_2, y_3]$ in each chamber.

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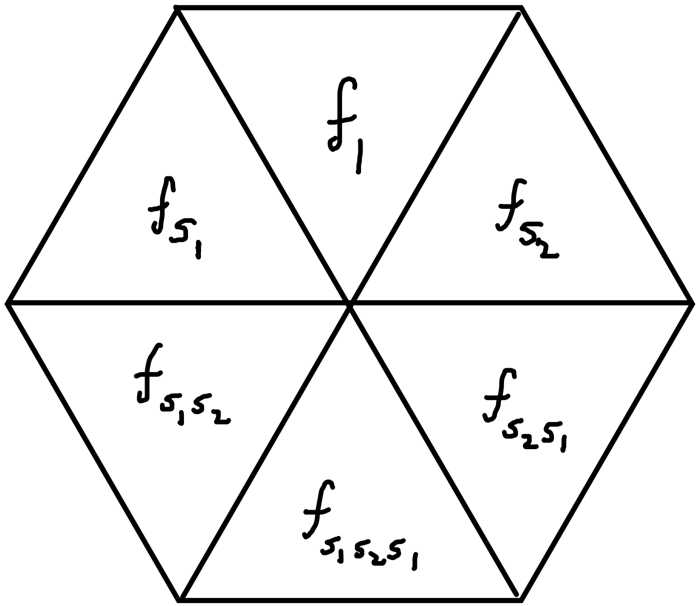
- $S \otimes S = \mathbb{C}[x_1, x_2, x_3, y_1, y_2, y_3]$ acts on $\bigoplus_{w \in W_0} \mathbb{C}[y_1, y_2, y_3]$




$$\in \bigoplus_{w \in W_0} \mathbb{C}[y_1, y_2, y_3]$$

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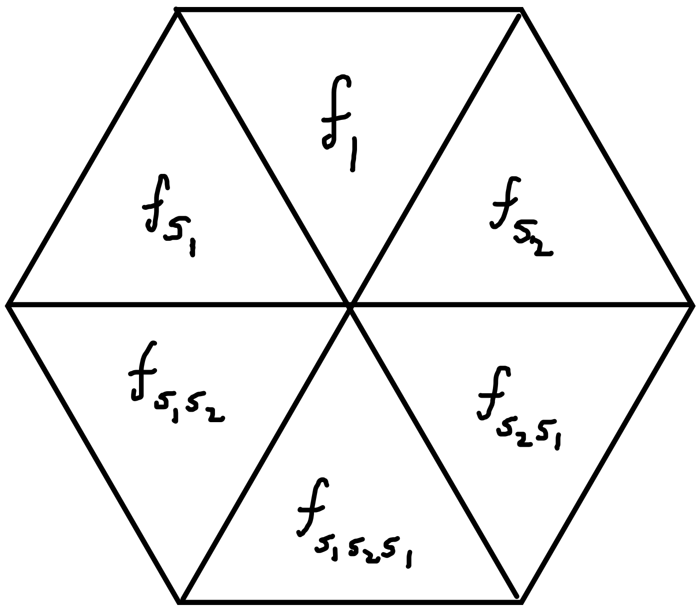


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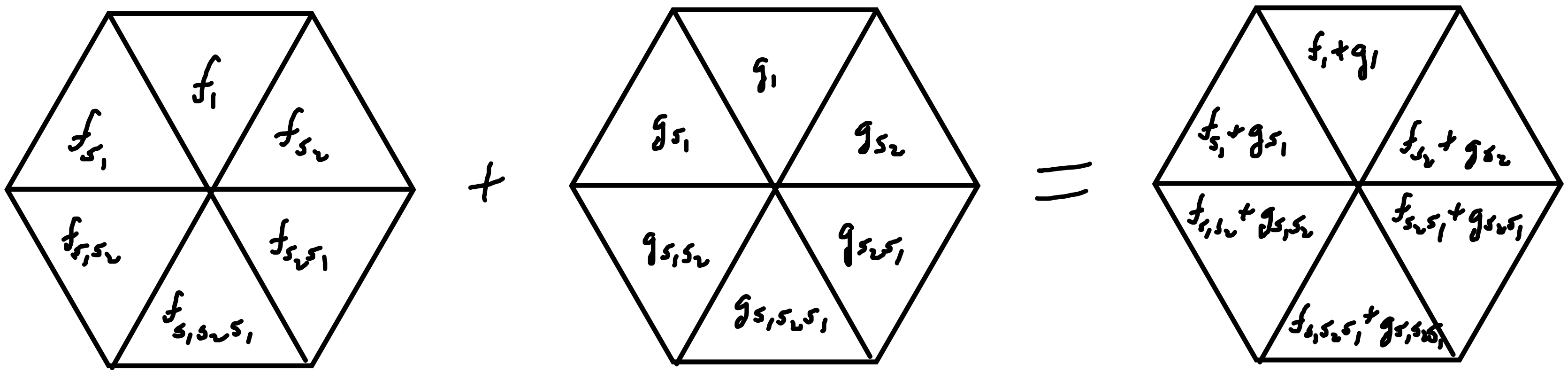
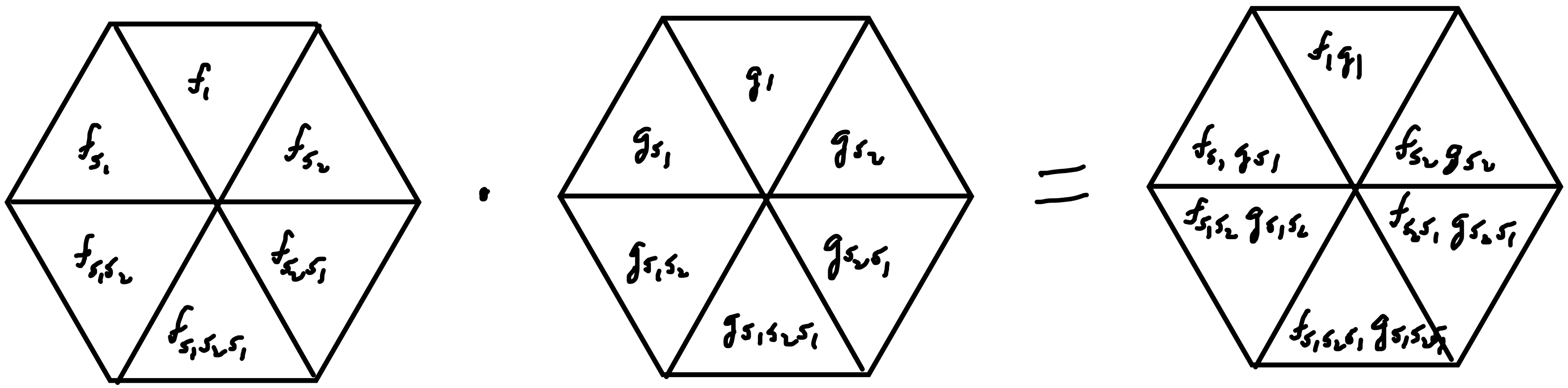
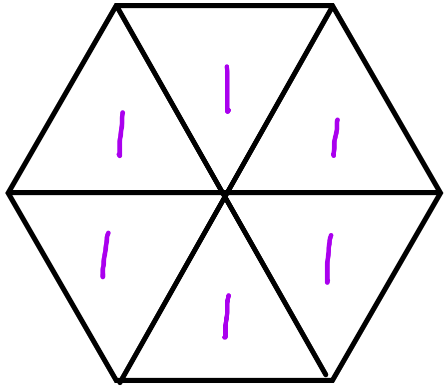

$f_1$
$f_{s_1}$
$f_{s_2}$
$f_{s_1 s_2}$
$f_{s_2 s_1}$
$f_{s_1 s_2 s_1}$
+
$g_1$
$g_{s_1}$
$g_{s_2}$
$g_{s_1 s_2}$
$g_{s_2 s_1}$
$g_{s_1 s_2 s_1}$
=
$f_1 + g_1$
$f_{s_1} + g_{s_1}$
$f_{s_2} + g_{s_2}$
$f_{s_1 s_2} + g_{s_1 s_2}$
$f_{s_2 s_1} + g_{s_2 s_1}$
$f_{s_1 s_2 s_1} + g_{s_1 s_2 s_1}$


$f_1$
$f_{s_1}$
$f_{s_2}$
$f_{s_1 s_2}$
$f_{s_2 s_1}$
$f_{s_1 s_2 s_1}$
$\cdot$
$g_1$
$g_{s_1}$
$g_{s_2}$
$g_{s_1 s_2}$
$g_{s_2 s_1}$
$g_{s_1 s_2 s_1}$
=
$f_1 g_1$
$f_{s_1} g_{s_1}$
$f_{s_2} g_{s_2}$
$f_{s_1 s_2} g_{s_1 s_2}$
$f_{s_2 s_1} g_{s_2 s_1}$
$f_{s_1 s_2 s_1} g_{s_1 s_2 s_1}$

$1 =$ 


$$y_{-\alpha_1} = y_2 - y_1 \qquad x_{-\alpha_1} = x_2 - x_1$$

$$y_{-\alpha_2} = y_3 - y_2 \qquad x_{-\alpha_2} = x_3 - x_2$$

$$y_{-(\alpha_1+\alpha_2)} = y_3 - y_1 \qquad x_{-(\alpha_1+\alpha_2)} = x_3 - x_1$$

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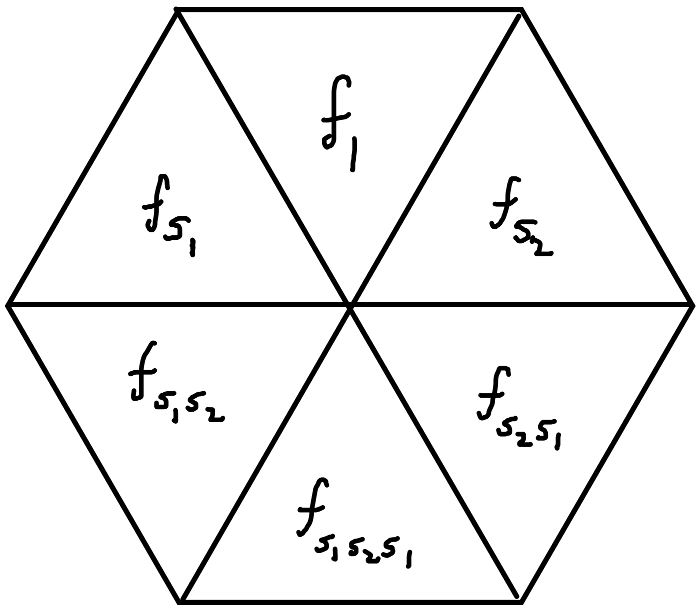

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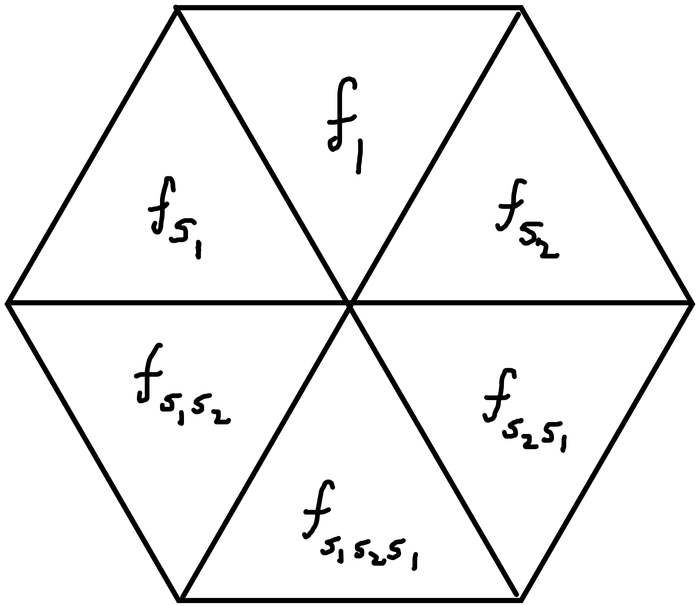
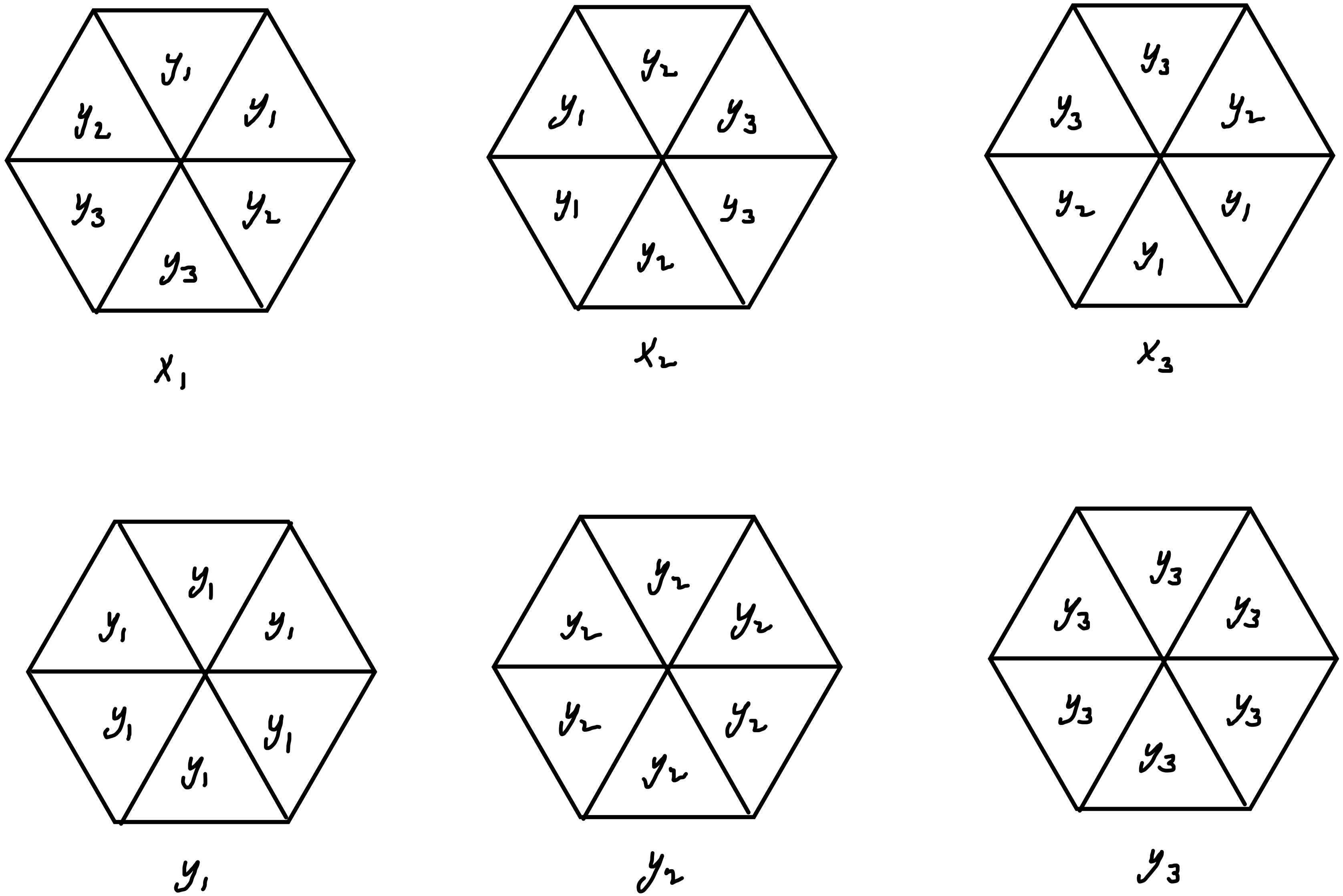



$$\in \bigoplus_{w \in W_0} \mathbb{C}[y_1, y_2, y_3]$$



$y_2$ $y_1$ $y_1$
$y_3$ $y_3$ $y_2$
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$y_1$ $y_2$ $y_3$
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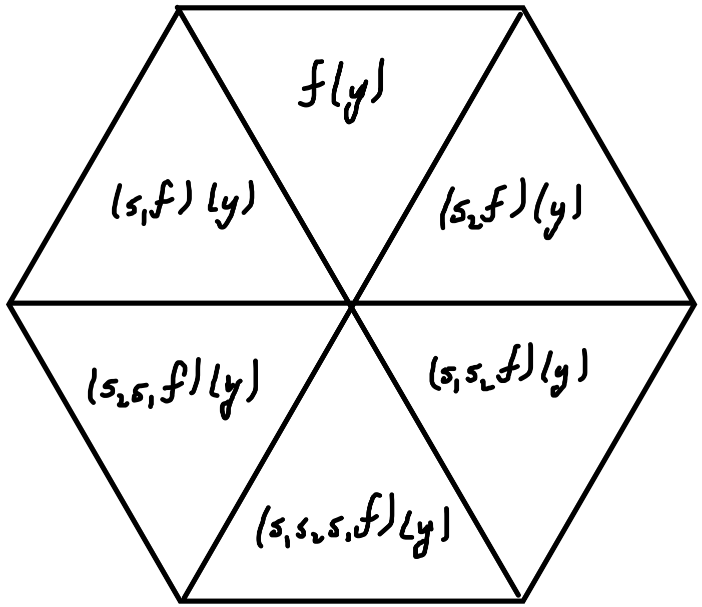
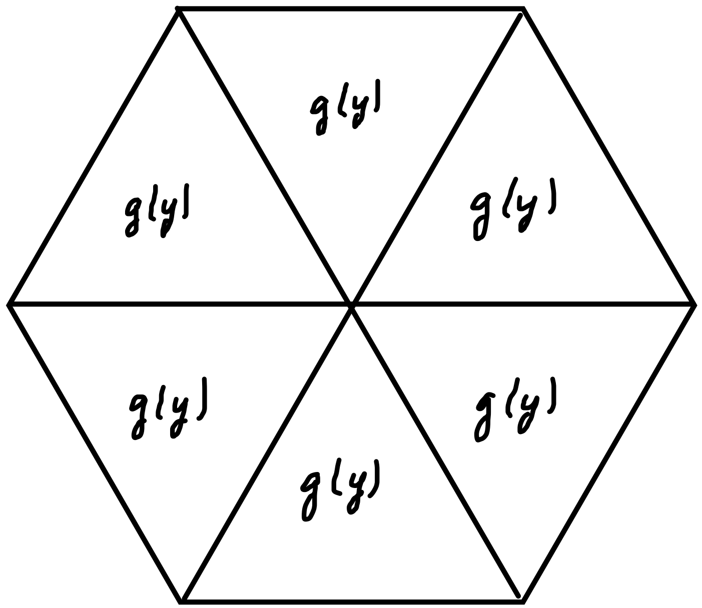
$y_3$

$$f(x_1, x_2, x_3) = f(x) =$$




$$g(y_1, y_2, y_3) = g(y) =$$



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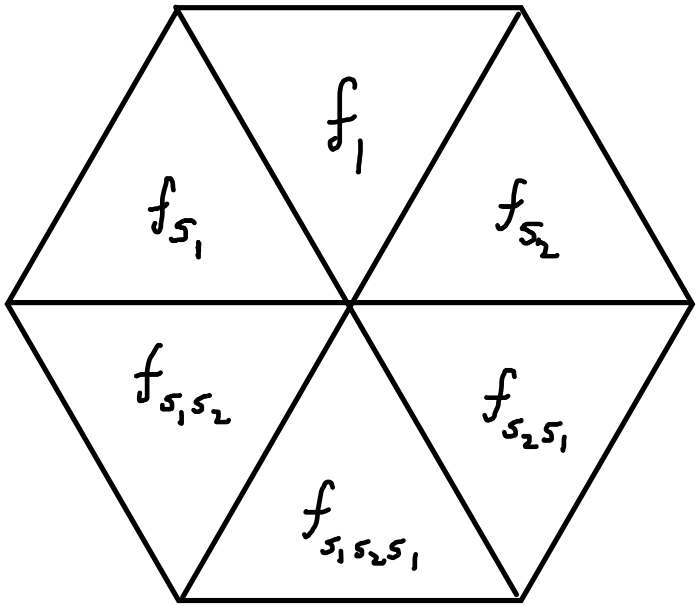


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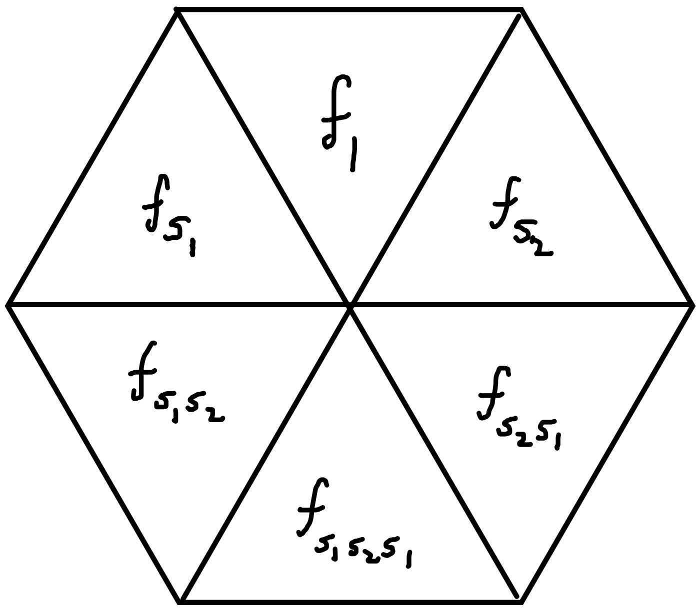


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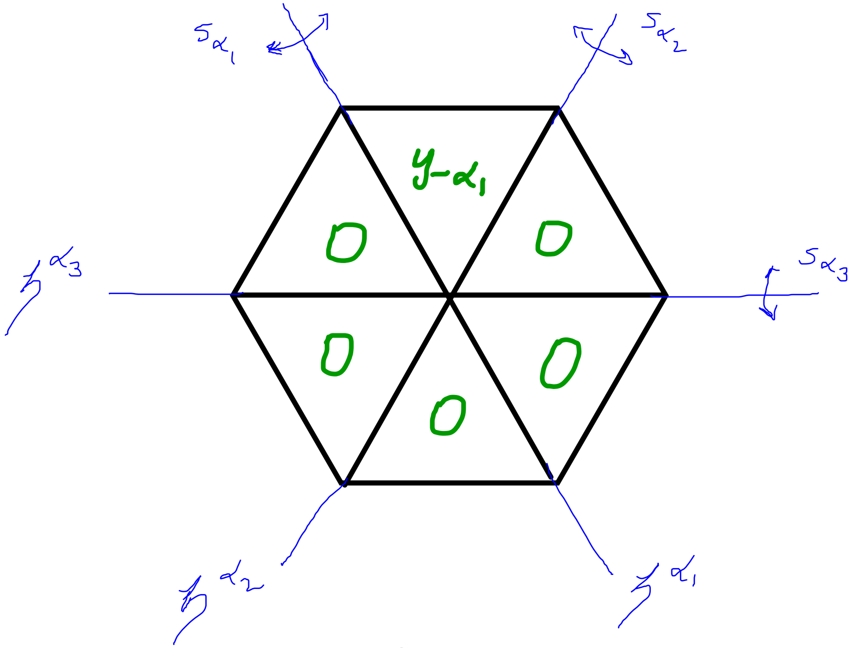



is an element of $\bigoplus_{w \in W_0} S$

that is not an element of

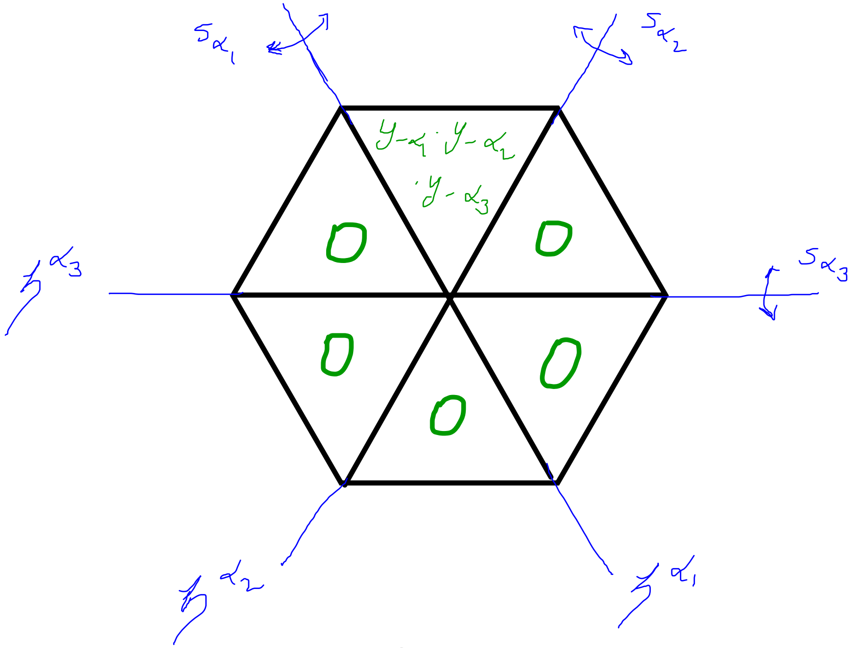
$$H_T(G/B) = (S \otimes S) \cdot 1 \subsetneq \bigoplus_{w \in W_0} S$$

$$H_T(G/B) = (S\otimes S)\cdot 1 \qquad \text{in } \bigoplus_{w\in W_0} H_T(pt)$$

<u>GKM Theorem</u>

For $\mathbb{Z}$-reflection groups $(W_0, \mathfrak{h}^*_{\mathbb{Z}})$

$$(S\otimes S)\cdot 1 = \left\{ (f_w)_{w\in W_0} \;\middle|\; f_w - f_{ws_\alpha} \in y_\alpha S \right\}$$

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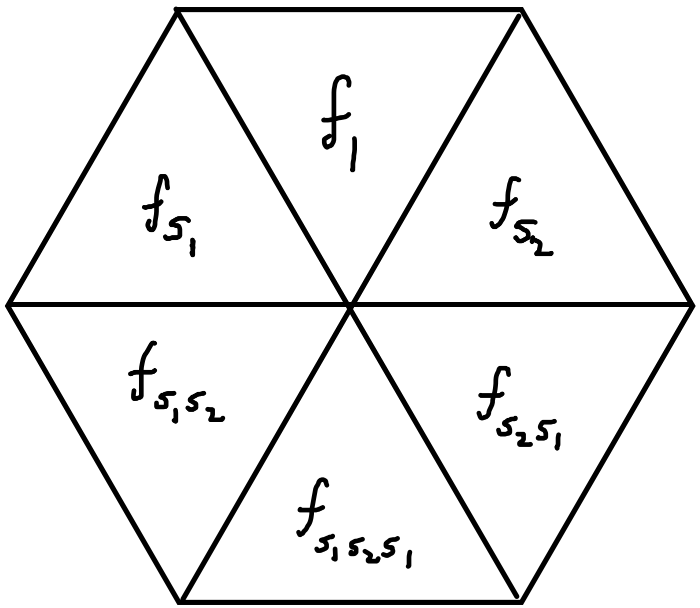


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**Borel presentation** $H_T(G/B) = S \otimes_{S^{W_0}} S$

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$\left\{\begin{matrix} BU \text{ } p\text{-compact group} \\ \uparrow \\ BD \text{ maximal } p\text{-compact torus} \end{matrix}\right\}$

$\updownarrow$ equivalence

$\left\{ \mathbb{Z}_p\text{-reflection groups } (W_0, \mathfrak{t}^*_{\mathbb{Z}_p}) \right\}$

$\mathcal{O}$-reflection groups $(W_0, \mathfrak{h}^*_{\mathcal{O}})$

A reflection is

$s \in GL_n(\mathbb{F})$ conjugate to $\begin{pmatrix} \xi & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$

An $\mathcal{O}$-reflection group $(W_0, \mathfrak{h}^*_{\mathcal{O}})$ is

- $\mathfrak{h}^*_{\mathcal{O}}$ is a free $\mathcal{O}$-module
- $W_0 \subseteq GL(\mathfrak{h}^*_{\mathcal{O}})$ a finite group generated by reflections

$S = S(\mathfrak{h}^*_{\mathcal{O}})$ is a free $S^{W_0}$-module.

# Cohomology of the flag variety

$$H_T(G/B)$$

Borel model: $H_T(G/B) = S \otimes_{S^{W_0}} S$

GKM model: $H_T(G/B) = (S \otimes S) \cdot 1$

# Cohomology of the flag variety

$$H_T(G/B)$$

generalize

generalize

- K-theory $K_T$
- Elliptic cohomology $Ell_T$
- Cobordism $\Omega_T$

done
- reductive algebraic groups

done
- compact Lie groups

done
- p-compact groups

Solution:

Change S

## Ordinary cohomology $S = H_T(pt)$

$$H_T(pt) = \mathbb{C}[y_\lambda \mid \lambda \in \mathfrak{h}^*_\mathbb{Z}] \quad \text{with} \quad y_{\lambda+\mu} = y_\lambda + y_\mu$$

## K-theory $S = K_T(pt)$

$$K_T(pt) = \mathbb{C}[y_\lambda \mid \lambda \in \mathfrak{h}^*_\mathbb{Z}] \quad \text{with} \quad y_{\lambda+\mu} = y_\lambda + y_\mu - y_\lambda y_\mu$$

## Elliptic cohomology $S = Ell_T(pt)$

$$Ell_T(pt) = \mathbb{C}[[y_\lambda \mid \lambda \in \mathfrak{h}^*_\mathbb{Z}]] \quad \text{with} \quad y_{\lambda+\mu} = y_\lambda + y_\mu - a_1 y_\lambda y_\mu - \cdots$$

## Cobordism $S = \Omega_T(pt)$

$$\Omega_T(pt) = \mathbb{C}[[y_\lambda \mid \lambda \in \mathfrak{h}^*_\mathbb{Z}]] \quad \text{with}$$

$$y_{\lambda+\mu} = y_\lambda + y_\mu + a_{11} y_\lambda y_\mu + a_{21} y_\lambda^2 y_\mu + a_{12} y_\lambda y_\mu^2 + a_{31} y_\lambda^3 y_\mu + a_{22} y_\lambda^2 y_\mu^2 + \cdots$$

where $a_{ij}$ satisfy relations so that

$$y_{\lambda+\mu} = y_{\mu+\lambda}, \qquad y_{(\lambda+\mu)+\nu} = y_{\lambda+(\mu+\nu)}, \qquad y_{\lambda+(-\lambda)} = y_0 = 0$$

# Ordinary cohomology $S = H_T(pt)$

$$H_T(pt) = S(\mathfrak{h}^*_{\mathbb{Z}}) = \mathbb{C}[y_1, \ldots, y_n] = \mathbb{C}[y_\lambda \mid \lambda \in \mathfrak{h}^*_{\mathbb{Z}}]$$

With

$$y_{\lambda+\mu} = y_\lambda + y_\mu$$

$$\mathfrak{h}^*_{\mathbb{Z}} = \mathbb{Z}\varepsilon_1 + \cdots + \mathbb{Z}\varepsilon_3$$

$y_1 = y_{\varepsilon_1}$

$y_2 = y_{\varepsilon_2}$

$y_3 = y_{\varepsilon_3}$

$$y_\lambda = y_{\lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n} = \underbrace{y_{\varepsilon_1} + \cdots + y_{\varepsilon_1}}_{\lambda_1} + \cdots + \underbrace{y_{\varepsilon_n} + \cdots + y_{\varepsilon_n}}_{\lambda_n}$$

## Ordinary cohomology $S = H_T(pt)$

$$H_T(pt) = \mathbb{C}[y_\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}^*] \quad \text{with} \quad y_{\lambda+\mu} = y_\lambda + y_\mu$$

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$$K_T(pt) = \mathbb{C}[y_\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}^*] \quad \text{with} \quad y_{\lambda+\mu} = y_\lambda + y_\mu - y_\lambda y_\mu$$

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## Cobordism $S = \Omega_T(pt)$

$$\Omega_T(pt) = \mathbb{C}[[y_\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}^*]] \quad \text{with}$$

$$y_{\lambda+\mu} = y_\lambda + y_\mu + a_{11} y_\lambda y_\mu + a_{21} y_\lambda^2 y_\mu + a_{12} y_\lambda y_\mu^2 + a_{31} y_\lambda^3 y_\mu + a_{22} y_\lambda^2 y_\mu^2 + \cdots$$

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$$y_{\lambda+\mu} = y_{\mu+\lambda}, \qquad y_{(\lambda+\mu)+\nu} = y_{\lambda+(\mu+\nu)}, \qquad y_{\lambda+(-\lambda)} = y_0 = 0$$

<u>Ordinary cohomology</u> $S = H_T(pt)$

$$H_T(pt) = S(\mathfrak{h}^*_{\mathbb{Z}}) = \mathbb{C}[y_\lambda \mid \lambda \in \mathfrak{h}^*_{\mathbb{Z}}]$$

with

$$y_{\lambda+\mu} = y_\lambda + y_\mu$$

<u>K-theory</u> $S = K_T(pt)$

$K_T(pt) = \mathbb{C}[y^\lambda \mid \lambda \in \mathfrak{h}^*_{\mathbb{Z}}]$ with $y^\lambda y^\mu = y^{\lambda+\mu}$

$K_T(pt) = \mathbb{C}[y_1^{\pm 1}, y_2^{\pm 1}, y_3^{\pm 1}]$ with $y^\lambda = y^{\lambda_1 \varepsilon_1 + \cdots + \lambda_n \varepsilon_n}$

$$= (y^{\varepsilon_1})^{\lambda_1} \cdots (y^{\varepsilon_n})^{\lambda_n}$$

$y_1 = y^{\varepsilon_1}, y_2 = y^{\varepsilon_2}, y_3 = y^{\varepsilon_3}$

$$= y_1^{\lambda_1} \cdots y_n^{\lambda_n}$$

## Ordinary cohomology $S = H_T(pt)$

$$H_T(pt) = S(\mathfrak{h}^*_{\mathbb{Z}}) = \mathbb{C}[y_\lambda \mid \lambda \in \mathfrak{h}^*_{\mathbb{Z}}]$$

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$$K_T(pt) = \mathbb{C}[y_\lambda \mid \lambda \in \mathfrak{h}^*_{\mathbb{Z}}] \quad \text{with}$$

$$y_\lambda = 1 - y^\lambda \qquad \text{so that} \quad y_{\lambda+\mu} = y_\lambda + y_\mu - y_\lambda y_\mu$$

## Ordinary cohomology $S = H_T(pt)$

$$H_T(pt) = \mathbb{C}[y_\lambda \mid \lambda \in \mathfrak{h}^*_{\mathbb{Z}}] \quad \text{with} \quad y_{\lambda+\mu} = y_\lambda + y_\mu$$

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## Elliptic cohomology $S = Ell_T(pt)$

$$Ell_T(pt) = \mathbb{C}[[y_\lambda \mid \lambda \in \mathfrak{h}^*_{\mathbb{Z}}]] \quad \text{with}$$

$$y_{\lambda+\mu} = y_\lambda + y_\mu - a_1 y_\lambda y_\mu - a_2 y_\lambda^2 y_\mu - a_2 y_\mu y_\lambda^2 - 2a_3 y_\lambda^3 y_\mu$$
$$- 2a_3 y_\lambda y_\mu^3 + (a_1 a_2 - 3a_3) y_\lambda^2 y_\mu^2 + \cdots$$

where the elliptic curve is

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

An alternative for Elliptic cohomology $S = Ell_T(pt)$

$$S = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} H^0(\mathbb{G}_\tau \otimes_{\mathbb{Z}} \mathfrak{h}^*_{\mathbb{Z}}, \mathcal{L}^{\otimes m})$$

abelian variety

ample line bundle

This ring also appears in the representation theory of affine Kac-Moody Lie algebras

$$S = \mathbb{C}[\hat{\mathfrak{h}}^*_{\mathbb{Z}}]^{\mathfrak{h}_{\mathbb{Z}}} \qquad S^{W_0} = Rep(\tilde{T})^{W_0 \ltimes \mathfrak{h}_{\mathbb{Z}}} = Rep(\hat{G})$$

An alternative for Elliptic cohomology $S = Ell_T(pt)$

$$S = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} H^0\left(\mathbb{G}_\tau \otimes_{\mathbb{Z}} \mathfrak{h}^*_{\mathbb{Z}}, \mathcal{L}^{\otimes m}\right)$$

ample line bundle

$$\mathbb{G}_\tau \otimes_{\mathbb{Z}} \mathfrak{h}^*_{\mathbb{Z}} = \mathfrak{h}^*_{\mathbb{C}} \big/ \mathfrak{h}^*_{\mathbb{Z}} + \tau \mathfrak{h}^*_{\mathbb{Z}}$$

if $\mathbb{G}_\tau = \dfrac{\mathbb{C}}{\mathbb{Z} + \tau\mathbb{Z}}$

This ring also appears in the representation theory of affine Kac-Moody Lie algebras

$$S = \mathbb{C}[\hat{\mathfrak{h}}^*_{\mathbb{Z}}]^{\mathfrak{h}_{\mathbb{Z}}}$$

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$$Ell_T(pt) = \mathbb{C}[[y_\lambda \mid \lambda \in \mathfrak{h}^*_{\mathbb{Z}}]] \quad \text{with} \quad y_{\lambda+\mu} = y_\lambda + y_\mu - a_1 y_\lambda y_\mu - \cdots$$

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# Cohomology of the flag variety

$$H_T(G/B)$$

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Change S

# Cohomology of the flag variety

$$H_T(G/B)$$

Borel model: $H_T(G/B) = S \otimes_{S^{W_0}} S$

GKM model: $H_T(G/B) = (S \otimes S) \cdot 1$

$$H_T(G/B) = (S\otimes S)\cdot 1 \quad \text{in } \bigoplus_{w\in W_0} H_T(pt)$$

<u>GKM theorem</u> For $\mathbb{Z}$-reflection groups $(W_0, \mathfrak{h}^*_{\mathbb{Z}})$.

$$(S\otimes S)\cdot 1 = \left\{ (f_w) \in \bigoplus_{w\in W_0} S \;\middle|\; f_w - f_{ws_\alpha} \in y_\alpha S \text{ for } w\in W_0,\ s_\alpha \text{ reflection} \right\}$$

(fixed hyperplane of $s_\alpha$ is $\mathfrak{h}^{s_\alpha} = \mathrm{span}\{\alpha\}$.

<u>Ortiz</u> For $\mathbb{Z}_p$-reflection groups $(W_0, \mathfrak{h}^*_{\mathbb{Z}_p})$

$$(S\otimes S)\cdot 1 = \left\{ (f_w) \in \bigoplus_{w\in W_0} S \;\middle|\; \sum_{j=0}^{r_s-1} \zeta^j f_{ws^j} \in y_\alpha^j S \right\}$$

$w\in W_0$, $s$ a reflection, $r_s$ = order of $r_s$

fixed hyperplane $\mathfrak{h}^s = \mathrm{span}\{\alpha\}$

$$s = \begin{pmatrix} \zeta & & 0 \\ & 1 & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}, \qquad \zeta = e^{\frac{2\pi i}{r_s}} \qquad (y_\alpha \text{ is } \alpha \text{ in } S = S(\mathfrak{h}^*))$$

# Cohomology of the flag variety

$$H_T(G/B)$$

Borel model: $H_T(G/B) = S \otimes_{S^{W_0}} S$

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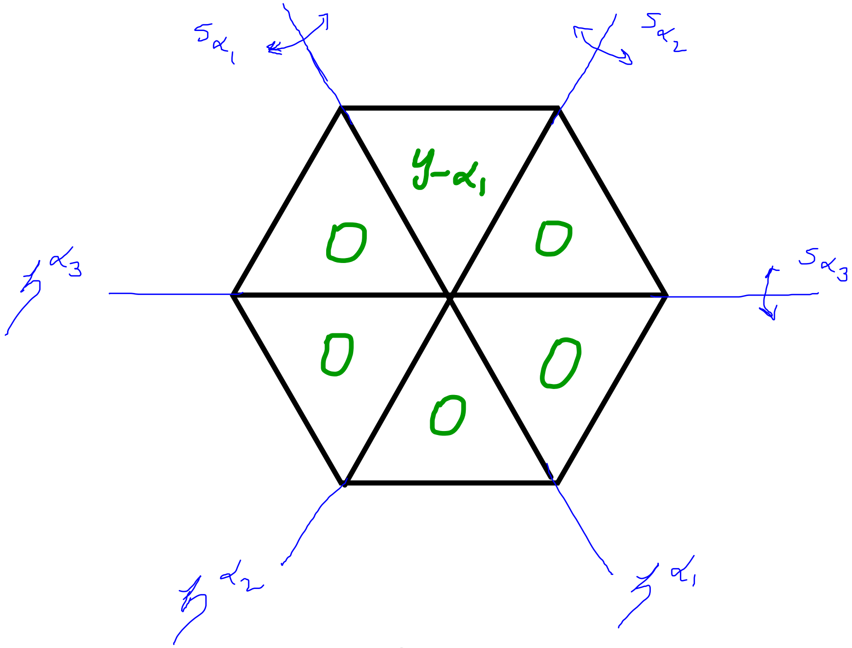



is an element of $\bigoplus_{w\in W_0} S$

that is not an element of

$$H_T(G/B) = (S\otimes S)\cdot 1 \subsetneq \bigoplus_{w\in W_0} S$$

$$H_T(G/B) = (S \otimes S) \cdot 1 \qquad \text{in } \bigoplus_{w \in W_0} H_T(pt)$$

<u>GKM Theorem</u>

For $\mathbb{Z}$-reflection groups $(W_0, \mathfrak{h}^*_{\mathbb{Z}})$

$$(S \otimes S) \cdot 1 = \left\{ (f_w)_{w \in W_0} \;\middle|\; f_w - f_{ws_\alpha} \in y_\alpha S \right\}$$

<u>Borel presentation</u> $H_T(G/B) = S \otimes_{S^{W_0}} S$

$$\left\{\begin{matrix} U \text{ compact Lie group} \\ \cup\!\!\mid \\ D \text{ maximal torus} \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} \mathbb{Z}\text{-reflection groups} \\ (W_0, \mathfrak{h}^*_{\mathbb{Z}}) \end{matrix}\right\}$$

$$\left\{\begin{matrix} BU \ p\text{-compact group} \\ \uparrow \\ BD \text{ maximal } p\text{-compact torus} \end{matrix}\right\}$$

$\updownarrow$ equivalence

$$\left\{ \mathbb{Z}_p\text{-reflection groups } (W_0, \mathfrak{h}^*_{\mathbb{Z}_p}) \right\}$$

$$H_T(G/B) = (S\otimes S)\cdot 1 \quad \text{in } \bigoplus_{w\in W_0} H_T(pt)$$

<u>GKM</u> <u>Theorem</u> For $\mathbb{Z}$-reflection groups $(W_0, \mathfrak{h}^*_{\mathbb{Z}})$

$$(S\otimes S)\cdot 1 = \left\{ (f_w)_{w\in W_0} \;\middle|\; f_w - f_{ws_\alpha} \in y_\alpha S \right\}$$

<u>Ortiz</u> For $\mathbb{Z}_p$-reflection groups $(W_0, \mathfrak{h}^*_{\mathbb{Z}_p})$

$$(S\otimes S)\cdot 1 = \left\{ (f_w)_{w\in W_0} \;\middle|\; \sum_{j=0}^{r_s-1} \zeta^j f_{ws^j} \in y_\alpha^j S \right\}$$

# $\mathcal{O}$-reflection groups $(W_0, \mathfrak{h}^*_{\mathcal{O}})$

A <u>reflection</u> is $s \in GL_n(\mathbb{F})$ conjugate to $\begin{pmatrix} \xi & & & 0 \\ & 1 & & \\ & & 1 & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix}$

An <u>$\mathcal{O}$-reflection group</u> $(W_0, \mathfrak{h}^*_{\mathcal{O}})$ is

- $\mathfrak{h}^*_{\mathcal{O}}$ is a free $\mathcal{O}$-module
- $W_0 \subseteq GL(\mathfrak{h}^*_{\mathcal{O}})$ a finite group generated by reflections

$S = S(\mathfrak{h}^*_{\mathcal{O}})$ is a free $S^{W_0}$-module.

# Pushing to a point: $G/B \xrightarrow{\pi} pt$

$\chi^\lambda : T \to \mathbb{C}^\times$ gives $\mathcal{L}_\lambda = G \times_B \mathbb{C}_\lambda$

homomorphism line bundle on $G/B$

$$\begin{array}{ccccc} \mathrm{Rep}(T) & \longrightarrow & H_T(G/B) & \longrightarrow & H_T(pt) = S \\ \chi^\lambda & \longmapsto & [\mathcal{L}_\lambda] & \longmapsto & \pi_!([\mathcal{L}_\lambda]) \end{array}$$

- For $H_T$: We get Weyl's dimension formula $\dim(L(\lambda))$
- For $K_T$: We get Weyl's character formula $\mathrm{char}(L(\lambda))$
- For $Ell_T$: We get Weyl-Kac chacter formula for the affine Lie algebra (Ganter arXiv:1206.0528)

$\mathrm{char}_{LT}(L(\lambda+m\delta))$ $\quad$ $\delta$ related to $\tau$ in $\mathbb{E}_\tau$

simple $\widehat{LG}$-module

$\delta$ keeps track of central extension $\widehat{LG}$

## Pushing to a point:

$$\pi : G/B \longrightarrow pt$$

$$\chi^\lambda : T \longrightarrow \mathbb{C}^\times \quad \text{gives} \quad \mathcal{L}_\lambda = G \times_B \mathbb{C}_\lambda$$

homomorphism — line bundle

$$\mathrm{Rep}(T) \longrightarrow H_T(G/B) \longrightarrow H_T(pt)$$

$$\chi^\lambda \longmapsto [\mathcal{L}_\lambda] \longmapsto \pi_!([\mathcal{L}_\lambda])$$

## Pushing to a point:

$$\pi_!([\mathcal{L}_\lambda])$$

- For $H_T$: Weyl's dimension formula $\dim(L(\lambda))$
- For $K_T$: Weyl's character formula $\mathrm{char}(L(\lambda))$
- For $Ell_T$: Weyl-Kac character formula for the loop group $LG$

Ganter arXiv:1206.0528

$\mathrm{char}_{LT}(L(\lambda+m\delta))$

simple $\widehat{LG}$-module

$\delta$ related to $\tau$ in $\mathbb{G}_\tau$, the elliptic curve

$\delta$ keeps track of central extension $\widehat{LG}$

# Cohomology of the flag variety

$$H_T(G/B)$$

Borel model: $H_T(G/B) = S \otimes_{S^{W_0}} S$

GKM model: $H_T(G/B) = (S \otimes S) \cdot 1$

# Schubert Calculus: Cohomology of the flag variety

$$[X_w] \in H_T(G/B)$$

**Linear algebra Theorem!**

$$G = GL_n(\mathbb{C}) \supseteq B = \left\{ \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \right\}$$

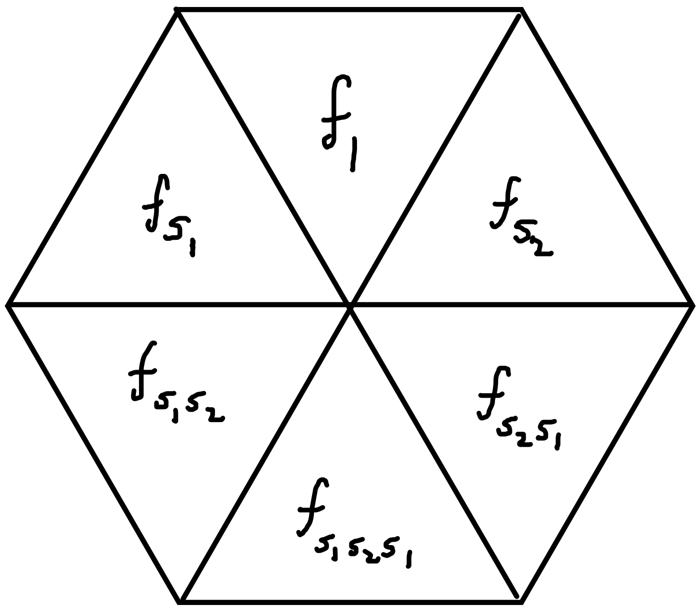
$$W_0 = S_n = \{ n \times n \text{ permutation matrices} \}$$

$$G = \bigsqcup_{w \in W_0} BwB$$

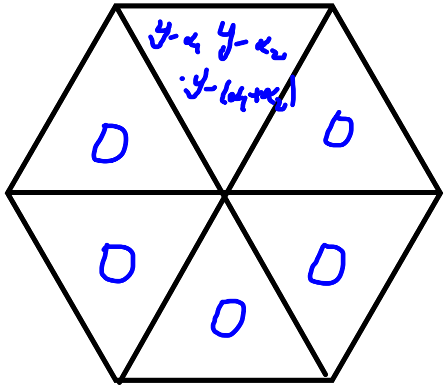
$X_w = \overline{BwB}$ are the Schubert Varieties

$$H_T(G/B) = (S \otimes S) \cdot 1 \qquad \text{in} \quad \bigoplus_{w \in W_0} S$$

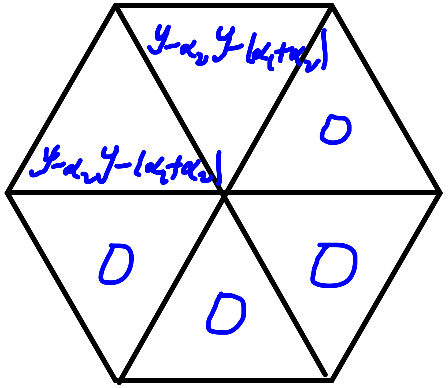
$$H_T(pt) = S = \mathbb{C}[y_\lambda \mid \lambda \in \mathfrak{h}^*_{\mathbb{Z}}] \quad \text{with} \quad y_{\lambda+\mu} = y_\lambda + y_\mu$$



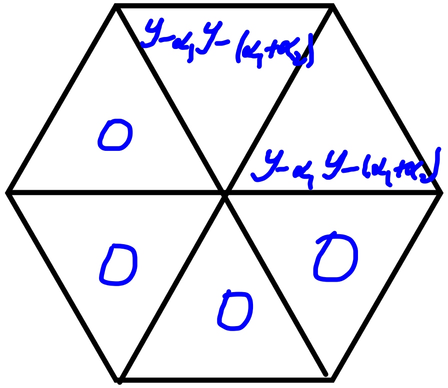

$$\in \quad \bigoplus_{w \in W_0} S$$



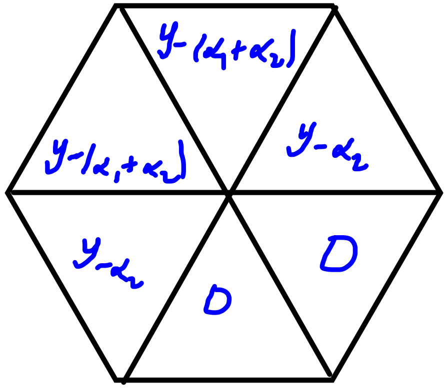

$[X_{pt}]$



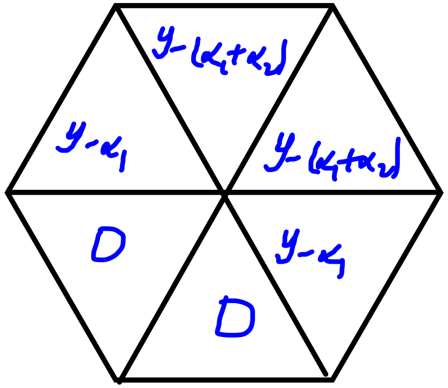

$[X_{s_1}]$



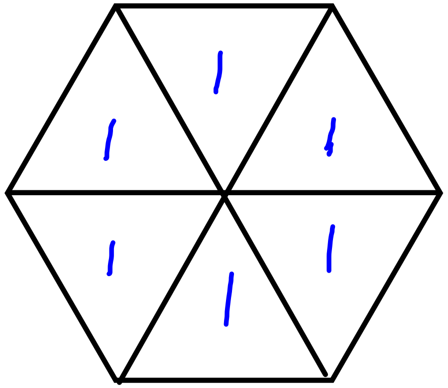

$[X_{s_2}]$




$[X_{s_1 s_2}]$




$[X_{s_2 s_1}]$




$[X_{s_1 s_2 s_1}]$

# Schubert Classes

**Ordinary cohomology** $S = H_T(pt)$

$$H_T(pt) = \mathbb{C}[y_\lambda \mid \lambda \in \mathfrak{h}^*_{\mathbb{Z}}] \quad \text{with} \quad y_{\lambda+\mu} = y_\lambda + y_\mu$$

**K-theory** $S = K_T(pt)$

$$K_T(pt) = \mathbb{C}[y_\lambda \mid \lambda \in \mathfrak{h}^*_{\mathbb{Z}}] \quad \text{with} \quad y_{\lambda+\mu} = y_\lambda + y_\mu - y_\lambda y_\mu$$

**Elliptic cohomology** $S = Ell_T(pt)$

$$Ell_T(pt) = \mathbb{C}[[y_\lambda \mid \lambda \in \mathfrak{h}^*_{\mathbb{Z}}]] \quad \text{with} \quad y_{\lambda+\mu} = y_\lambda + y_\mu - a_1 y_\lambda y_\mu - \cdots$$

**Cobordism** $S = \Omega_T(pt)$

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$$y_{\lambda+\mu} = y_\lambda + y_\mu + a_{11} y_\lambda y_\mu + a_{21} y_\lambda^2 y_\mu + a_{12} y_\lambda y_\mu^2 + a_{31} y_\lambda^3 y_\mu + a_{22} y_\lambda^2 y_\mu^2 + \cdots$$

where $a_{ij}$ satisfy relations so that

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**Conjecture** Cobordism Schubert Classes

There exist unique $[X_w]$, $w \in W_0$ characterized by

(a) $\{[X_w] \mid w \in W_0\}$ is a basis of $\Omega_T(G/B)$,

(b) $[X_w]_w = \prod_{\substack{\alpha \in R^+ \\ w\alpha \notin R^+}} y_{-\alpha}$ and $[X_w]_v = 0$ unless $v \le w$.

(c) If $\lambda$ is a dominant weight $\left(\lambda \in \sum_{i=1}^{n} \mathbb{Z}_{\geq 0} \omega_i\right)$

$$X_{-\lambda}[X_w] = \sum_{v \in W_0} c_{\lambda w}^{v} [X_v]$$

with $c_{\lambda w}^{v} \in \mathbb{L}_{\geq 0}[y_{-\alpha_1}, \ldots, y_{-\alpha_n}]$ where $\mathbb{L}_{\geq 0} = \mathbb{Z}_{\geq 0}[a_{11}, a_{12}, a_{21}, \ldots]$.

## ~~Conjecture~~ Wrong Cobordism Schubert Classes

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## Wrong Conjecture   Cobordism Schubert Classes

There exist unique $[X_w]$, $w \in W_0$ characterized by

(a) $\{[X_w] \mid w \in W_0\}$ is a basis of $\Omega_T(G/B)$,

(b) $[X_w]_w = \prod_{\substack{\alpha \in R^+ \\ w\alpha \notin R^+}} y_{-\alpha}$ and $[X_w]_v = 0$ unless $v \le w$.

(c) If $\lambda$ is a dominant weight $\left(\lambda \in \sum_{i=1}^{n} \mathbb{Z}_{\ge 0}\omega_i\right)$

$$X_{-\lambda}[X_w] = \sum_{v \in W_0} c_{\lambda w}^{v} [X_v]$$

with $c_{\lambda w}^{v} \in \mathbb{L}_{\ge 0}[y_{-\alpha_1}, \ldots, y_{-\alpha_n}]$ where $\mathbb{L}_{\ge 0} = \mathbb{Z}_{\ge 0}[a_{11}, a_{12}, a_{21}, \ldots]$.

POSITIVITY

Wrong but good

# Conjecture Cobordism Schubert Classes

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POSITIVITY

# More Conjectures Cobordism Schubert Classes

(a) The $[X_w]$ satisfy

$$[X_u][X_v] = \sum_{w\in W_0} c_{uv}^{w}\,[X_w]$$

with $c_{\lambda w}^{v} \in \mathbb{L}_{\geq 0}[y_{-\alpha_1},\dots,y_{-\alpha_n}]$ where $\mathbb{L}_{\geq 0} = \mathbb{Z}_{\geq 0}[a_{11},a_{12},a_{21},\dots]$.

(b) If $w_0$ is the longest element of $W_0$ and $s_i$ is a simple reflection then

$$[X_{w_0 s_i}] = x_{\omega_i} \oplus y_{w_0\omega_i}$$

Note: Already $X_{w_0 s_i}$ has singularities.

## Cohomology of the flag variety

Borel model: $H_T(G/B) = S \otimes_{S^{W_0}} S$

GKM model: $H_T(G/B) = (S \otimes S) \cdot 1$

- done • K-theory $K_T$
- done • Elliptic cohomology $Ell_T$
- done • Cobordism $\Omega_T$

- done • reductive algebraic groups
- done • compact Lie groups
- done • p-compact groups

Ortiz: GKM criterion for the p-compact setting

Ganter: $G/B \to pt$ gives characters of $G$ and $LG$ modules

You: Explain what $[X_w]$ is for cobordism $\Omega_T(G/B)$