Combinatorics, Representations and Elliptic Cohomology ~~What is a~~ Schur function? Colloquium, Stanford University 09.05.2013

$W_0 = S_3$ acts on $S = \mathbb{C}[y_1^{\pm 1}, y_2^{\pm 1}, y_3^{\pm 1}] = \mathbb{C}[\mathfrak{h}_{\mathbb{Z}}]$ by permuting y_1, y_2, y_3. Let $y^\lambda = y_1^{\lambda_1} y_2^{\lambda_2} y_3^{\lambda_3}$

$$\begin{array}{ccccc} \mathbb{C}[\mathfrak{h}_{\mathbb{Z}}]^{W_0} = u_0\mathbb{C}[\mathfrak{h}_{\mathbb{Z}}] & \xrightarrow[\sim]{a_\rho \cdot} & e_0\mathbb{C}[\mathfrak{h}_{\mathbb{Z}}] \\ m_\lambda = u_0 y^\lambda & & \\ s_\lambda & \longleftarrow\!\shortmid & e_0 y^{\lambda+\rho} = a_{\lambda+\rho} \end{array}$$

where $u_0 = \sum_{w \in W_0} w$ and $e_0 = \sum_{w \in W_0} \mathrm{sgn}(w) w$

$$a_\rho = (y_1 - y_2)(y_1 - y_3)(y_2 - y_3) = \prod_{1 \le i < j \le 3} (y_i - y_j) = \det\begin{pmatrix} 1 & y_1 & y_1^2 \\ 1 & y_2 & y_2^2 \\ 1 & y_3 & y_3^2 \end{pmatrix}$$

This is a story about

$$\begin{array}{ccccc} G^\vee & \supseteq & B^\vee & \supseteq & T^\vee \\ \| & & \| & & \| \\ GL_3(\mathbb{C}) & \supseteq & \left\{\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}\right\} & \supseteq & \left\{\begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}\right\} \end{array}$$

$$\mathfrak{h}_{\mathbb{Z}} = \mathrm{Hom}(T^\vee, \mathbb{C}^\times) = \{\text{irreducible } T^\vee\text{-representations}\}$$
$$= \mathbb{Z}\text{-span}\{\varepsilon_1, \varepsilon_2, \varepsilon_3\} = \mathbb{Z}^3 \subseteq \mathbb{R}^3$$

where

$$\begin{array}{rcl} y_i = y^{\varepsilon_i} : T^\vee & \longrightarrow & \mathbb{C}^\times \\ \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix} & \longmapsto & t_i \end{array}$$

②

Hermann-Weyl

Irreducible ~~repre~~ $G^\vee$-modules $L(\lambda)$ have

$$s_\lambda = \mathrm{Res}^{G^\vee}_{T^\vee}(L(\lambda)) = \bigoplus_{\mu\in\mathfrak{h}_\mathbb{Z}} (y^\mu)^{\oplus K_{\lambda\mu}}$$

so

$$s_\lambda = \sum_{\mu\in\mathfrak{h}_\mathbb{Z}} K_{\lambda\mu} y^\mu = \sum_{\mu\in\mathfrak{h}_\mathbb{Z}/W_0} K_{\lambda\mu} m_\mu \quad \text{with}$$

$$K_{\lambda\mu} = \dim(L(\lambda)_\mu) \in \mathbb{Z}_{\geq 0}.$$

Macdonald polynomials

$\mathbb{C}[\mathfrak{h}_\mathbb{Z}]$ has two bases

$$\{y^\lambda \mid \lambda\in\mathfrak{h}_\mathbb{Z}\} \quad \text{and} \quad \{E_\lambda(q,t) \mid \lambda\in\mathfrak{h}_\mathbb{Z}\}$$

Then

$$\mathbb{C}[\mathfrak{h}_\mathbb{Z}]^{W_0} = \mathbb{1}_0\,\mathbb{C}[\mathfrak{h}_\mathbb{Z}] \xrightarrow[\sim]{A_\rho\,\cdot} \varepsilon_0\,\mathbb{C}[\mathfrak{h}_\mathbb{Z}]$$

$$P_\lambda(q,t) \quad = \mathbb{1}_0 E_\lambda(q,t)$$

$$P_\lambda(q,qt) \longleftarrow\!\!\!\shortmid \ \varepsilon_0 E_{\lambda+\rho}(q,t) = A_{\lambda+\rho}(q,t)$$

Define $K_{\lambda\mu}(q,t)$ by

$$P_\lambda(q,qt) = \sum_\mu K_{\lambda\mu}(q,t)\, P_\mu(q,t)$$

At $q=0, t=1$, this is $s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$.

$q=0,\ t=\frac{1}{p}$

$$\mathbb{C}[Y_{\mathfrak{a}}]^{W_0} \xrightarrow{\mathbb{1}_0} \mathbb{1}_0 H \mathbb{1}_0 \xrightarrow{A_\rho} \varepsilon_0 H \mathbb{1}_0$$

$$P_\lambda(0,t) \longleftarrow \mathbb{1}_0 Y^\lambda \mathbb{1}_0$$

$$s_\lambda \longleftarrow \varepsilon_0 Y^{\lambda+\rho}\mathbb{1}_0 = A_{\lambda+\rho}(0,t)$$

$P_\lambda(0,t)$ is a spherical function for $GL_3(\mathbb{Q}_p)$

$A_{\lambda+\rho}(0,t)$ is a Whittaker vector for $GL_3(\mathbb{Q}_p)$

$$G = GL_3(\mathbb{Q}_p) = GL_3(\mathbb{F}_p((\varpi))^{\text{carries}})$$

$$\cup\!|$$

$$K = GL_3(\mathbb{Z}_p) = GL_3(\mathbb{F}_p[[\varpi]]^{\text{carries}}) \xrightarrow[\Phi]{\varpi=0} GL_3(\mathbb{F}_p)$$

$$\cup\!| \qquad\qquad\qquad\qquad\qquad\qquad \cup\!|$$

$$I = \Phi^{-1}(B) \longrightarrow B = \left\{\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}\right\}$$

$H = C_c(I\backslash G/I)$ and $H\mathbb{1}_0 = C_c(I\backslash G/K)$

as $\mathbb{1}_0 =$ char. function of K, $\mathbb{1}_0(g) = \begin{cases} 1, & \text{if } g\in K \\ 0, & \text{if } g\notin K \end{cases}$

$\mathbb{1}_0 H \mathbb{1}_0 = C_c(K\backslash G/K) = K_0(\mathrm{Perv}_K(G/K))$

$\mathbb{1}_0 Y^\lambda \mathbb{1}_0 =$ char. fun. of $K t_\lambda K$

Brylinski-Kostant Define $K_{\lambda\mu}(t)$ by

$$s_\lambda = \sum_\mu K_{\lambda\mu}(t) P_\mu(0,t)$$

Then

$$K_{\lambda\mu}(t) = \sum_i t^i \dim \left(L(\lambda)_\mu^{(i+1)} \big/ L(\lambda)_\mu^{(i)} \right) \in \mathbb{Z}_{\geq 0}[t]$$

where

$$L(\lambda)_\mu^{(i)} = \left\{ m \in L(\lambda) \,\middle|\, \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}^i m = 0 \right\}.$$

Cohomology of a pt = the ring S

Ordinary cohomology = G_a-cohomology $\quad h_{T^\vee} = H_{T^\vee}$

$S = H_{T^\vee}(pt) = S(\mathfrak{h}_{\mathbb{Z}}) = \mathbb{C}[y_1, y_2, y_3]$

$\quad = \mathbb{C}[y_\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}]$ with $y_{\lambda+\mu} = y_\lambda + y_\mu$

K-theory = G_m-cohomology $\quad h_{T^\vee} = K_{T^\vee}$

$S = K_{T^\vee}(pt) = Rep(T^\vee) = \mathbb{C}[y_1^{\pm 1}, y_2^{\pm 1}, y_3^{\pm 1}]$

$\quad = \mathbb{C}[y^\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}]$ with $y^{\lambda+\mu} = y^\lambda y^\mu$

$\quad = \mathbb{C}[y_\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}]$ with $y_{\lambda+\mu} = y_\lambda + y_\mu - y_\lambda y_\mu$

setting $y_\lambda = 1 - y^\lambda$.

<u>Elliptic cohomology</u> = $\mathcal{E}_\tau$-cohomology $\quad h_{T^\vee} = \mathcal{E}_{T^\vee}$

$S = \mathcal{E}_{T^\vee}(pt) = \mathbb{C}[[y_\lambda \mid \lambda \in \mathfrak{h}^*_{\mathbb{R}}]]$ with

$$y_{\lambda+\mu} = y_\lambda + y_\mu - a_1 y_\lambda y_\mu - a_2 y_\lambda^2 y_\mu - a_2 y_\lambda y_\mu^2 - 2a_3 y_\lambda^3 y_\mu$$
$$- 2a_3 y_\lambda y_\mu^3 + (a_1 a_2 - 3a_3) y_\lambda^2 y_\mu^2 + \cdots$$

if the elliptic curve $\mathcal{E}_\tau$ is

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

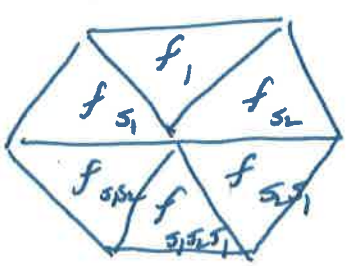
Put these all in one package with $h_{T^\vee} = \Omega_{T^\vee}$,

$$y_{\lambda+\mu} = y_\lambda + y_\mu + a_{11} y_\lambda y_\mu + a_{21} y_\lambda^2 y_\mu + a_{12} y_\lambda y_\mu^2 + \cdots$$

<u>Cohomology of $G^\vee/B^\vee$</u>

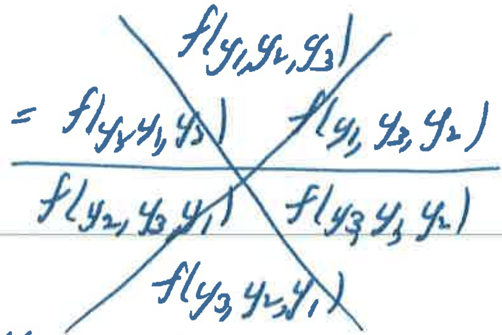
$$h_{T^\vee}(G^\vee/B^\vee) = (S \otimes S)\cdot 1.$$

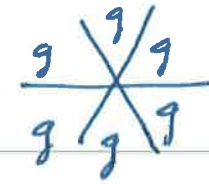
Think of $S = \mathbb{C}[y_1, y_2, y_3]$ and $W_0 = \langle s_1, s_2 \mid s_i^2 = 1,\ s_1 s_2 s_1 = s_2 s_1 s_2 \rangle$



$\in \bigoplus_{w \in W_0} S$ and $1 =$

$S \otimes S = \mathbb{C}[x_1, x_2, x_3, y_1, y_2, y_3]$ acts on $\bigoplus_{w \in W_0} S$

$f(x_1, x_2, x_3) =$ 

and $g(y_1, y_2, y_3) = g =$ 

We Study $G^\vee/B^\vee$ with

$$G^\vee = \bigsqcup_{w \in W_0} B^\vee w B^\vee \quad \text{and}$$

$$X_w = \overline{B^\vee w B^\vee} = \bigsqcup_{v \le w} B^\vee v B^\vee$$ are the <u>Schubert varieties</u>

<u>Conjecture</u> there exist unique

$[X_w]$, $w \in W_0$ in $\Omega_{\mathcal{F}}(G^\vee/B^\vee)$

characterized by

(a) $[X_w]_w = \prod_{\substack{\alpha \in R^+ \\ w\alpha \in R^+}} y_{-\alpha}$ and $[X_w]_v = 0$ unless $v \le w$

(b) If $\lambda = \lambda_1\varepsilon_1 + \lambda_2\varepsilon_2 + \lambda_3\varepsilon_3 \in \mathfrak{h}_{\mathbb{Z}}^*$ is dominant $(\lambda_1 \ge \lambda_2 \ge \lambda_3)$

$$x_\lambda [X_w] = \sum_{v \in W_0} c_{\lambda w}^{v} [X_v]$$

with $c_{\lambda w}^{v} \in \mathbb{Z}_{\ge 0}[a_{ij}][[y_{-\omega_1}, y_{-\omega_2}, y_{-\omega_3}]]$

$\left(\omega_i = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_i \text{ and } R^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le 3\} \right)$