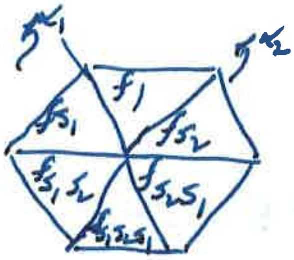
Colloquium, University of Oregon, 06.05.2013

Elliptic cohomology h_T of the flag variety G/B

$$h_T(G/B) = (S\otimes S)\cdot 1$$

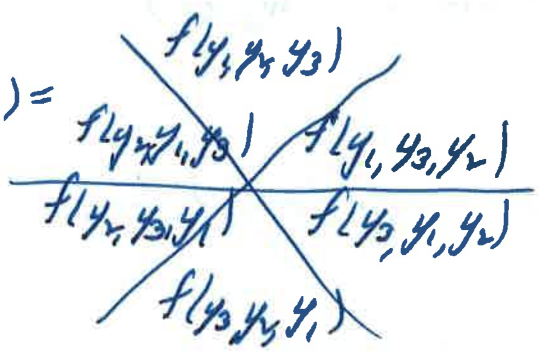
$S=\mathbb{C}[y_1,y_2,y_3]$ and $W_0=\langle s_1,s_2 \mid s_i^2=1,\ s_1s_2s_1=s_2s_1s_2\rangle$



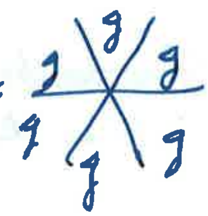
$\in \bigoplus_{w\in W_0} S$ and $1 =$

$S\otimes S = \mathbb{C}[x_1,x_2,x_3,y_1,y_2,y_3]$ acts on $\bigoplus_{w\in W_0} S$ by

$f(x_1,x_2,x_3)=$



and $g(y_1,y_2,y_3)=g=$



$$G = GL_3(\mathbb{C})$$
$$\cup\!|$$
$$B = \left\{\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}\right\} \quad \text{and} \quad G = \bigsqcup_{w\in W_0} BwB$$
$$\cup\!|$$
$$T = \left\{\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}\right\}$$

$X_w = \overline{BwB} = \bigsqcup_{v\leq w} BvB$ are the <u>Schubert varieties</u>

4 rings = 4 cohomologies h_T

$$\mathfrak{h}_{\mathbb{Z}}^* = \mathrm{Hom}(T, \mathbb{C}^\times) = \mathbb{Z}\text{-span}\{\varepsilon_1, \varepsilon_2, \varepsilon_3\} = \mathbb{Z}^3 \subseteq \mathbb{R}^3$$

Ordinary cohomology = $\mathbb{G}_a$-cohomology

$S = S(\mathfrak{h}_{\mathbb{Z}}^*) = H_T(pt) = \mathbb{C}[y_1, y_2, y_3]$ $\quad (y_i = y_{\varepsilon_i})$

$= \mathbb{C}[y_\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}^*]$ with $y_{\lambda+\mu} = y_\lambda + y_\mu$

K-Theory = $\mathbb{G}_m$-cohomology

$S = K_T(pt) = \mathbb{C}[\mathfrak{h}_{\mathbb{Z}}^*] = \mathbb{C}[Y^\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}^*]$ with $Y^{\lambda+\mu} = Y^\lambda Y^\mu$

$= \mathbb{C}[Y_1^{\pm 1}, Y_2^{\pm 1}, Y_3^{\pm 1}] = \mathbb{C}[y_\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}^*]$ with $y_{\lambda+\mu} = y_\lambda + y_\mu - y_\lambda y_\mu$

Elliptic cohomology = $\mathbb{G}_E$-cohomology.

If the elliptic curve $\mathbb{G}_E$ is

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6 \quad \text{then}$$

$S = \mathbb{C}[[y_\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}^*]]$ with

$$y_{\lambda+\mu} = y_\lambda + y_\mu - a_1 y_\lambda y_\mu - a_2 y_\lambda^2 y_\mu - a_2 y_\lambda y_\mu^2 - 2a_3 y_\lambda^3 y_\mu$$
$$- 2a_3 y_\lambda y_\mu^3 + (a_1 a_2 - 3a_3) y_\lambda^2 y_\mu^2 + \cdots$$

To cover all cases at once: $h_T = \Omega_T$

$$y_{\lambda+\mu} = y_\lambda + y_\mu + a_{11} y_\lambda y_\mu + a_{21} y_\lambda^2 y_\mu + a_{12} y_\lambda y_\mu^2 + \cdots$$

Conjecture There exist unique

$$[X_w],\ w \in W_0, \qquad \text{in } h_T(G/B)$$

characterized by

(a) $[X_w]_w = \prod_{\substack{\alpha \in R^+ \\ w\alpha \in R^+}} y_{-\alpha}$ and $[X_w]_v = 0$ unless $v \le w$,

(b) If $\lambda \in \mathfrak{h}_{\mathbb{Z}}^*$ is dominant $(\lambda = \lambda_1\varepsilon_1 + \lambda_2\varepsilon_2 + \lambda_3\varepsilon_3,\ \lambda_1 \ge \lambda_2 \ge \lambda_3)$

$$x_{-\lambda}[X_w] = \sum_{v \in W_0} c_{\lambda v}^{w} [X_v]$$

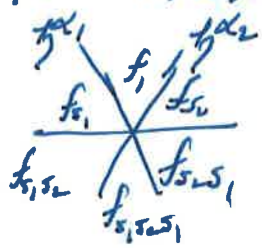
with $c_{\lambda v}^{w} \in \mathbb{Z}_{\ge 0}[a_{ij}][[y_{-\omega_1}, y_{-\omega_2}, y_{-\omega_3}]]$

$(\omega_i = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_i$ and $R^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le n\})$

Note: If $w = s_{i_1} \cdots s_{i_\ell}$ and

$[Z_{i_1 \cdots i_\ell}] = [P_{i_1} \times_B P_{i_2} \times_B \cdots \times_B P_{i_\ell} \times_B pt \xrightarrow{\gamma_{i_1 \cdots i_\ell}} X_w \subseteq G/B]$ then

$[Z_{i_1 \cdots i_\ell}] = A_{i_1} \cdots A_{i_\ell} [X_1]$ where $A_i = (1 + t_{s_i}) \dfrac{1}{x_{-\alpha_i}}$

t_{s_i} acts on  by flipping in $\mathfrak{h}^{\alpha_i}$.

If $h_T = H_T$ or $h_T = K_T$ and $w = s_{i_1} \cdots s_{i_\ell}$ is reduced

$$[Z_{i_1 \cdots i_\ell}] = X_w$$

BUT $[Z_{121}] \ne [X_{s_1s_2s_1}]$ if $h_T = E_T$ or $h_T = \Omega_T$.

Is $S = E_T(pt)$ tractable?

Let $P \supseteq B$ be a subgroup of G,

$\tau \in \mathbb{C}$ and $\Lambda = \mathbb{Z}\text{-span}\{\gamma_1, \ldots, \gamma_{2n}\} \subseteq \mathbb{C}^n$

P is a parabolic subgroup if G/P is a projective variety

$E_\tau = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ is an elliptic curve if E_τ is a projective variety

$A_\Lambda = \mathbb{C}^n/\Lambda$ is an abelian variety if A_Λ is a projective variety

Rewriting $S = E_T(pt)$

$S = \mathbb{C}[[y_\lambda \mid \lambda \in \mathfrak{h}_\mathbb{Z}^*]]$ with $y_{\lambda+\mu} = y_\lambda + y_\mu + a_{11} y_\lambda y_\mu + \cdots$

$S = \mathcal{O}_{E_\tau \otimes_\mathbb{Z} \mathfrak{h}_\mathbb{Z}^*}$, structure sheaf of $E_\tau \otimes_\mathbb{Z} \mathfrak{h}_\mathbb{Z}^* = \dfrac{\mathfrak{h}_\mathbb{C}^*}{\mathfrak{h}_\mathbb{Z} + \tau \mathfrak{h}_\mathbb{Z}}$

$S = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} H^0(E_\tau \otimes_\mathbb{Z} \mathfrak{h}_\mathbb{Z}^*, \mathcal{L}^{\otimes m})$ homogeneous coord. ring

$= \mathbb{C}\text{-span}\{\Theta_{\lambda + m\mathbb{Z}} \mid \lambda \in \mathfrak{h}_\mathbb{Z}^* \bmod m\mathfrak{h}_\mathbb{Z},\ m \in \mathbb{Z}_{\geq 0}\}$

$= \mathbb{C}[\hat{\mathfrak{h}}_\mathbb{Z}^*]^{\mathfrak{h}_\mathbb{Z}} = Rep(\hat{\mathfrak{g}})^{\mathfrak{h}_\mathbb{Z}}$

POINT: Use Representation Theory to compute with S.

$y^\lambda \in Rep(T) = \mathbb{C}[\mathfrak{h}^*_{\mathbb{Z}}] = K_T(pt)$

$$\begin{array}{c} \cap\! \mid \\ \mathbb{C}[\mathfrak{h}^*_{\mathbb{Z}}]^{W_0} = u_0\, \mathbb{C}[\mathfrak{h}^*_{\mathbb{Z}}] \xrightarrow{a_\rho \cdot} e_0\, \mathbb{C}[\mathfrak{h}^*_{\mathbb{Z}}] \end{array}$$

$\underbrace{Res^G_T(H^0(G/B, \mathcal{L}_\lambda))}_{\text{simple } G\text{-module } M_\lambda} = s_\lambda \longleftarrow e_0\, y^{\lambda+\rho} = a_{\lambda+\rho}$

$= u_0 y^\lambda$

where $u_0 = \sum_{w \in W_0} w$ and $e_0 = \sum_{w \in W_0} \mathrm{sgn}(w)\, w$

$$a_\rho = (y_1 - y_2)(y_1 - y_3)(y_2 - y_3) = \det\begin{pmatrix} 1 & y_1 & y_1^2 \\ 1 & y_2 & y_2^2 \\ 1 & y_3 & y_3^2 \end{pmatrix}$$

Now go to

$$LG = \text{loop group} = \{S^1 \xrightarrow{\gamma} GL_3\} = GL_3(\mathbb{C}((t)))$$

$$\uparrow$$

$$\hat{G}' = \text{central extension of } LG$$

$$\cap\!\mid$$

$\hat{T} \subseteq \hat{G}$ = affine Kac-Moody group

then $\hat{\mathfrak{h}}^*_{\mathbb{Z}} = Hom(\hat{T}, \mathbb{C}^\times) = \mathfrak{h}^*_{\mathbb{Z}} \oplus \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\delta$ and

$Rep(\hat{T}) = \mathbb{C}[\hat{\mathfrak{h}}^*_{\mathbb{Z}}]$

$\cap\!\mid$

$S = \mathbb{C}[\hat{\mathfrak{h}}^*_{\mathbb{Z}}]^{\mathfrak{h}_{\mathbb{Z}}}$

$\cap\!\mid$

$Rep(\hat{G}) = \mathbb{C}[\hat{\mathfrak{h}}^*_{\mathbb{Z}}]^{W} = (\mathbb{C}[\hat{\mathfrak{h}}^*_{\mathbb{Z}}]^{\mathfrak{h}_{\mathbb{Z}}})^{W_0} = u_0\, \mathbb{C}[\hat{\mathfrak{h}}^*_{\mathbb{Z}}]^{\mathfrak{h}_{\mathbb{Z}}} \xrightarrow{\hat{a}_\rho} e_0\, \mathbb{C}[\hat{\mathfrak{h}}^*_{\mathbb{Z}}]^{\mathfrak{h}_{\mathbb{Z}}}$

Char. of Irred. rep of $\hat{G}$ $\quad \hat{s}_\lambda \longleftarrow e_0\, \Theta_{\lambda+\rho} = \hat{a}_{\lambda+\rho}$