Connecting Macdonald, Whittaker and Demazure Symmetric functions

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$W_0 = S_n$ acts on $\mathbb{C}[\mathfrak{h}_\mathbb{Z}] = \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$

by permuting $X_1, \ldots, X_n$. Let $X^\lambda = X_1^{\lambda_1} \cdots X_n^{\lambda_n}$. $\left(\begin{array}{l} \mathfrak{h}_\mathbb{Z} = \mathbb{Z}^n \\ \lambda \mapsto (\lambda_1, \ldots, \lambda_n) \end{array}\right)$

$$\mathbb{C}[\mathfrak{h}_\mathbb{Z}]^{W_0} = \{ f \in \mathbb{C}[\mathfrak{h}_\mathbb{Z}] \mid wf = f \text{ for } w \in W_0 \}.$$

$q=q$ and $t=t$: Macdonald

$$\mathbb{C}[\mathfrak{h}_\mathbb{Z}]^{W_0} = \mathbb{1}_0 \mathbb{C}[\mathfrak{h}_\mathbb{Z}] \xrightarrow[\sim]{A_\rho \cdot} \varepsilon_0 \mathbb{C}[\mathfrak{h}_\mathbb{Z}]$$

$$P_\lambda(q, qt) \longleftarrow\!\mid \varepsilon_0 E_{\lambda+\rho}(q,t) = A_{\lambda+\rho}(q,t)$$

$$P_\lambda(q,t) = \mathbb{1}_0 E_\lambda(q,t)$$

$q=0$ and $t=1$: Hermann Weyl

$$\mathbb{C}[\mathfrak{h}_\mathbb{Z}]^{W_0} = u_0 \mathbb{C}[\mathfrak{h}_\mathbb{Z}] \xrightarrow{a_\rho} e_0 \mathbb{C}[\mathfrak{h}_\mathbb{Z}]$$

$$s_\lambda \longleftarrow\!\mid e_0 X^{\lambda+\rho} = a_{\lambda+\rho}$$

$$m_\lambda = u_0 X^\lambda$$

$q=0$ and $t=\frac{1}{p}$: Local Langlands

$$\mathbb{C}[\mathfrak{h}_\mathbb{Z}]^{W_0} \xrightarrow{\sim} \mathbb{1}_0 H \mathbb{1}_0 \xrightarrow{A_\rho \cdot} \varepsilon_0 H \mathbb{1}_0$$

$$s_\lambda \longleftarrow\!\mid \varepsilon_0 X^{\lambda+\rho} \mathbb{1}_0$$

$$P_\lambda(0,t) \longleftarrow\!\mid \mathbb{1}_0 X^\lambda \mathbb{1}_0$$

"Polynomials"

$\mathbb{C}[\mathfrak{h}_\mathbb{Z}]$ has favorite bases

$\{X^\lambda \mid \lambda \in \mathfrak{h}_\mathbb{Z}\}$ and $\{E_\lambda(q,t) \mid \lambda \in \mathfrak{h}_\mathbb{Z}\}$

with $E_\lambda(q,t) = X^\lambda +$ lower terms

At $q=0$ and $t=1$:

$$u_0 = \sum_{w \in W_0} w \quad \text{and} \quad e_0 = \sum_{w \in W_0} \det(w) w$$

$$a_\rho = \prod_{1 \le i < j \le n} (X_i - X_j) = \det \begin{pmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ 1 & x_2 & \cdots & x_2^{n-1} \\ \vdots & & & \vdots \\ 1 & x_n & \cdots & x_n^{n-1} \end{pmatrix}$$

At $q=0$ and $t=\frac{1}{p}$

$$\begin{array}{lll} G = G(\mathbb{Q}_p) = G(\mathbb{F}_p((\varpi))^{\text{carries}}) & & \\ \cup\!\vert & & \\ K = G(\mathbb{Z}_p) = G(\mathbb{F}_p[[\varpi]]^{\text{carries}}) & \longrightarrow & G(\mathbb{F}_p) \\ \cup\!\vert & & \cup\!\vert \\ I = \Phi^{-1}(B) & \longrightarrow & B = \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\} \end{array}$$

$H = C_c(I \backslash G / I) \qquad H\mathbf{1}_0 = C_c(I \backslash G / K) \qquad \mathbf{1}_0 H \mathbf{1}_0 = C_c(K \backslash G / K)$

where $\mathbf{1}_0 =$ char. function of K.

$$G = \bigsqcup_{w \in W_0 \ltimes \mathfrak{h}_\mathbb{Z}} IwI = \bigsqcup_{\lambda \in \mathfrak{h}_\mathbb{Z}} I t_\lambda K = \bigsqcup_{\lambda \in \mathfrak{h}_\mathbb{Z}/W_0} K t_\lambda K.$$

Weight multiplicities: The Langlands dual groups

$$G^\vee = GL_n(\mathbb{C}) \supseteq T^\vee = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}$$

$\mathbb{C}[\mathfrak{h}_\mathbb{Z}] = Rep(T)$: Irreducible $T^\vee$-modules X^λ

$$X^\lambda : T^\vee \to \mathbb{C}^\times \quad \text{are indexed by } \lambda \in \mathfrak{h}_\mathbb{Z}$$

$\mathbb{C}[\mathfrak{h}_\mathbb{Z}]^{W_0} = Rep(G)$: Irreducible $G^\vee$-modules $L(\lambda)$

$$s_\lambda = Res^{G^\vee}_{T^\vee}(L(\lambda)) \simeq \bigoplus_{\mu \in \mathfrak{h}_\mathbb{Z}} (X^\mu)^{\oplus K_{\lambda\mu}}$$

Write

$$s_\lambda = \sum_{\mu \in \mathfrak{h}_\mathbb{Z}/W_0} K_{\lambda\mu} m_\mu,$$

$$s_\lambda = \sum_{\mu \in \mathfrak{h}_\mathbb{Z}/W_0} K_{\lambda\mu}(t) P_\mu(0,t),$$

$$P_\lambda(q,qt) = \sum_{\mu \in \mathfrak{h}_\mathbb{Z}/W_0} K_{\lambda\mu}(q,t) P_\mu(q,t),$$

We know

$$K_{\lambda\mu} = \dim(L(\lambda)_\mu), \qquad K_{\lambda\mu}(t) = \sum_i t^i \dim\left(\frac{L(\lambda)_\mu^{(i+1)}}{L(\lambda)_\mu^{(i)}} \right)$$

$$\text{where } L(\lambda)_\mu^i = \left\{ m \in L(\lambda)_\mu \;\middle|\; \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}^i m = 0 \right\}$$

Conjecture: $K_{\lambda\mu}(q,t) \in \mathbb{Z}_{\geq 0}[q,t] \cdot \frac{1}{e_{\lambda\mu}(q,t)}$ with $e_{\lambda\mu}(q,t) \in \mathbb{Z}[q,t]$ and $e_{\lambda\mu}(0,t) = 1$

Using H to get information about G-modules

$$H = C_c(I\backslash G/I) \text{ has basis } \{T_w \mid w \in W_0 \ltimes \mathfrak{h}_{\mathbb{Z}}\}$$

where T_w = char. function of IwI.

If $M \xrightarrow{\mathcal{N}} N$ is a G-module morphism

then $M^I \xrightarrow{\mathcal{N}^I} N^I$ is an H-module morphism.

The <u>universal unramified principal series</u> is

$$\mathcal{B} = \left\{ f: G \to \mathbb{C} \;\middle|\; \begin{array}{l} f(mug) = f(g) \text{ for } m \in \mathring{T}, u \in U^+, g \in G \\ f \text{ is locally constant} \\ \mathrm{supp}(f) \subseteq \mathring{T}U^+C_f \text{ for a compact } C_f \subseteq G \end{array} \right\}$$

Then

$\mathcal{B}^I$ has basis $\{X^v \mid v \in W_0 \ltimes \mathfrak{h}_{\mathbb{Z}}\}$

where X^v = char. fcn. of $\mathring{T}U^+vI$.

As right H-modules,

$$\begin{array}{ccc} H = C_c(I\backslash G/I) & \xrightarrow[\sim]{X^!} & C_c(\mathring{T}U^+\backslash G/I) = \mathcal{B}^I = X^! \cdot H \\ T_w & \longmapsto & X^! T_w \\ T_{i_1}^{\epsilon_1} \cdots T_{i_\ell}^{\epsilon_\ell} & \longleftarrow\!\shortmid & X^v \end{array}$$

So

$$\mathbb{C}[Y/\mathbb{Z}]\mathbb{1} = H\mathbb{1}_0 = C_c(I\backslash G/K) \xrightarrow[\sim]{X^!} C_c(\mathring{T}U^+\backslash G/K) = \mathcal{B}^K = X^! H\mathbb{1}_0$$

Two natural operators on $\mathbb{C}[Y/\mathbb{Z}]\mathbb{1}$.

$$T_i X^\lambda = X^{s_i\lambda} t^{\frac{1}{2}} + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \frac{X^\lambda - X^{s_i\lambda}}{1 - X^{-\alpha_i}}$$

$$= \left(t^{\frac{1}{2}} s_i + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \Delta_i\right) X^\lambda .$$

and

$$\tau_i X^\lambda = X^{s_i\lambda} \tau_i \qquad \left(\begin{array}{l} \text{in } \tilde{H}, \\ \tau_i = T_i - \dfrac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - y^{\alpha_i}} \end{array} \right)$$

<u>Ion's important points:</u>

① τ_i^{qt} at $q = \infty$ is T_i.

② τ_i^{qt} at $t = \infty$ is a Demazure operator

③ $\overline{E_\lambda(\infty, t)} = E_\lambda(0, t)$ where $\overline{} : H\mathbb{1}_0 \to H\mathbb{1}_0$ is the Kazhdan-Lusztig bar involution.