For the group $G_2$:

$$G_2(R) = \text{Aut}_{R-alg}(D \otimes R)$$

where

$$D = \mathbb{Z}\text{-span}\{ e_0, e_1, \ldots, e_7, e_0^2 + e_1^2 + e_2^2 + e_3^2 + e_4^2, e_0 e_1 + e_2 e_3 + e_4 e_5 + e_6 e_7, e_0 e_2 + e_1 e_3 + e_4 e_6 + e_5 e_7, e_0 e_3 + e_1 e_2 + e_4 e_5 + e_6 e_7, e_0 e_4 + e_1 e_5 + e_2 e_6 + e_3 e_7, e_0 e_5 + e_1 e_6 + e_2 e_7 + e_3 e_4, e_0 e_6 + e_1 e_7 + e_2 e_4 + e_3 e_5, e_0 e_7 + e_1 e_4 + e_2 e_5 + e_3 e_6 \}$$

Recall that if $u \in D$ with $u = s_0 e_0 + \cdots + s_7 e_7$

$$\overline{u} = s_0 e_0 - s_1 e_1 - \cdots - s_7 e_7, \quad Tr(u) = u + \overline{u} = 2 s_0$$

$$N(u) = uu^* = s_0^2 + s_1^2 + \cdots + s_7^2, \quad \langle x, y \rangle = Tr(x^* y)$$

Let $k$ be a field, char$(k) \neq 2$. Let

$$M = D \otimes k \quad \text{and} \quad M_0 = k\{ e_0^2 \} \quad \langle x, e_0 \rangle = 0.$$  

**Fact 1:** If $x \in M_0$ then $N(x) = -x^2.$

**Fact 2:** $M_0 = \mathbb{F} x \mathbb{E} M | x \in k$ and $x^2 \in k.$

$$= \mathbb{F} x \mathbb{E} M | x \in k \quad \text{and} \quad x^2 \in k.$$  

**Fact 3:** If $g \in \text{Aut}(D \otimes k) = G_2(k)$ then $g M_0 \subseteq M_0.$

**Fact 4:** $M_0$ is the 7-dimensional irreducible $G_2(k)$-module.

**Fact 5:** If $g \in G_2(k)$ and $x \in M$ then

$$g x = g(x), \quad N(gx) = N(x), \quad Tr(gx) = Tr(x)$$  

since $x$ is characterized by $x + \overline{x} \in k$ and $x - \overline{x} \in M_0.$
A null line is \( \mathbf{V} = \text{span} \{ \mathbf{v} \} \) with \( \mathbf{v} \in \mathbf{V} \) and \( N(\mathbf{v}) = 0 \).

A null plane is \( \mathbf{V} = \text{span} \{ \mathbf{v}_1, \mathbf{v}_2 \} \) with \( \mathbf{x} \cdot \mathbf{y} = 0 \) for \( \mathbf{x}, \mathbf{y} \in \mathbf{V} \).

Let

\[
\mathbf{V}_1 = k \mathbf{v}_1, \text{ where } \mathbf{v}_1 \in \mathcal{M}_0 \text{ with } N(\mathbf{v}_1) = -\mathbf{v}_1^2 = 0
\]

\[
\mathbf{V}_2 = k \mathbf{v}_1 + k \mathbf{v}_2, \text{ where } \mathbf{v}_1 \in \mathcal{M}_0 \text{ with } \mathbf{v}_1 \cdot \mathbf{v}_2 = 0.
\]

\[
\mathbf{V}_3 = \{ \mathbf{y} \in \mathcal{M}_0 \mid \mathbf{x} \cdot \mathbf{y} = 0 \text{ for } \mathbf{x} \in \mathbf{V}_2 \}
\]

\[
\mathbf{V}_4 = \mathbf{V}_3^\perp
\]

\[
\mathbf{V}_5 = \mathbf{V}_2^\perp
\]

\[
\mathbf{V}_6 = \mathbf{V}_1^\perp
\]

\[
\mathbf{V}_7 = \mathcal{M}_0
\]

so that \( 0 = \mathbf{V}_1 \leq \mathbf{V}_2 \leq \mathbf{V}_3 \leq \mathbf{V}_4 \leq \mathbf{V}_5 \leq \mathbf{V}_6 \leq \mathcal{M}_0 \)

is determined by the data of \( \mathbf{V}_1 \leq \mathbf{V}_2 \).

Define

\[
\mathcal{P}_1 = \text{Stab} (\mathbf{V}_1), \quad \mathcal{P}_2 = \text{Stab} (\mathbf{V}_2)
\]

\[
\mathcal{B} = \text{Stab} (\mathbf{V}_1 \leq \mathbf{V}_2) = \text{Stab} (\mathbf{V}_1 \leq \mathbf{V}_2 \leq \ldots \leq \mathbf{V}_6 \leq \mathcal{M}_0)
\]

\( \mathcal{B} \) is a Borel subgroup of \( G_2(k) \)

and \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) are two examples of standard parabolic subgroups.
$G_2$ has class number 1 so that

$$G_2(\hat{\mathbb{A}}) = G(\mathbb{Q}) G(\hat{\mathbb{A}})$$

in general, if $G$ is defined over $\mathbb{Q}$

$$G(\hat{\mathbb{A}}) = G(\mathbb{Q}) G(\hat{\mathbb{A}}) \cup \ldots \cup G(\mathbb{Q}) G(\hat{\mathbb{A}})$$

and $h$ is the class number of $G$.

Let $G$ be a connected reductive algebraic group over $\mathbb{Q}$

which is anisotropic over $\mathbb{R}$,

$W$ a $G(\mathbb{Q})$-module,

$K$ an open compact subgroup of $G(\hat{\mathbb{A}})$.

The space of algebraic modular forms of weight $W$

and level $K$ on $G$ is

$$M(W, K) = \{ F : G(\hat{\mathbb{A}})/K \to W \mid F(\delta g) = \delta F(g) \text{ for } \delta \in G(\mathbb{Q}) \}$$

Then there

$$K = \prod_p K_p \quad \text{where} \quad J_p = \{ G(\mathbb{Z}_p) \mid \text{parahoric, } \delta \notin S \}$$

so that

$$G(\mathbb{Z}_p) \xrightarrow{q = 0} G(\mathbb{F}_p)$$

$$U_1 \quad U_1$$

and

$$J_p = \mathfrak{F}(\mathfrak{p}) \to \mathfrak{p}$$

$$G(\mathbb{Z}_p) / J_p \to G(\mathbb{F}_p) / \mathfrak{p}.$$
Goal: Study the Hecke algebra action on $M/W, K$.

WHAT Hecke algebra? WHAT action?

Recall:

let $G$ be a finite group, $H_i$ and $H_i'$ subgroups of $G$.
let $W_i$ be a representation of $H_i$, $W_i : H_i \rightarrow GL(V_i) = GL_n(\mathbb{C})$.
let $W_i'$ be a representation of $H_i'$, $W_i' : H_i' \rightarrow GL(V_i') = GL_{n'}(\mathbb{C})$.

Define

$M = \{ f : G \rightarrow \text{Hom}(V_i, V_i') | f(h_i h_i') = W_i(h_i) f(h_i') W_i'(h_i') \} \forall h_i \in H_i, h_i' \in H_i'$.

Then

$\text{Hom}_G(\text{Ind}^G_{H_i}(W_i), \text{Ind}^G_{H_i'}(W_i')) \cong M$ as vector spaces.

Special case:

$\text{Hom}_G(\text{Ind}^G_{H_i}(W_i), \text{Ind}^G_{H_i'}(W_i'))$ is an algebra, and
$\text{Hom}_G(\text{Ind}^G_{H_i}(\text{triv}), \text{Ind}^G_{H_i'}(\text{triv}))$ is the Hecke algebra of the pair $(G, W_i)$.

Then $M$ is a left $\text{End}_G(\text{Ind}^G_{H_i}(W_i))$-module.

and a right $\text{End}_G(\text{Ind}^G_{H_i'}(W_i'))$-module.

so $M/W, K)$ is a left $\text{End}_G(\text{Ind}^G_{H_i}(W_i))$-module.

and a right $\text{End}_G(\text{Ind}^G_{H_i'}(\text{triv}))$-module.

$\text{End}_G(\text{Ind}^G_{H_i'}(\text{triv}))$ is the Hecke algebra.