The octonions and G_2 Number Theory seminar 7 November 2012
Univ. Melbourne.

The octonions $\mathbb{O}$

The octonions is the $\mathbb{R}$-algebra (nonassociative)

$$\mathbb{O} = \mathbb{R}\text{-span}\{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$$

with product determined by

$$(x_1, y_1)(x_2, y_2) = (x_1 x_2 - \bar{y}_2 y_1,\ y_1 \bar{x}_2 + y_2 x_1)$$

with

$$1 = (1,0),\quad e_1 = (i,0),\quad e_2 = (j,0),\quad e_3 = (k,0)$$

$$e_4 = (0,e)\quad e_5 = (0,i)\quad e_6 = (0,j)\quad e_7 = (0,k)$$

where $\mathbb{H} = \mathbb{R}\text{-span}\{1, i, j, k\}$ is the algebra of quaternions.

The conjugation $\bar{\ }: \mathbb{O} \to \mathbb{O}$, trace $Tr: \mathbb{O} \to \mathbb{R}$ and norm $N: \mathbb{O} \to \mathbb{R}$ are given by

$$\bar{u} = \xi_0 e_0 - \xi_1 e_1 - \xi_2 e_2 - \xi_3 e_3 - \xi_4 e_4 - \xi_5 e_5 - \xi_6 e_6 - \xi_7 e_7$$

$$Tr(u) = u + \bar{u} = 2\xi_0$$

$$N(u) = \xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 + \xi_5^2 + \xi_6^2 + \xi_7^2$$

for $u = \xi_0 e_0 + \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 + \xi_4 e_4 + \xi_5 e_5 + \xi_6 e_6 + \xi_7 e_7$

HW: Show that, if $u \in \mathbb{O}$ and $u \neq 0$ then u is invertible with inverse

$$u^{-1} = \frac{1}{N(u)} \bar{u}$$

HW: Show that if $x, y \in \mathbb{O}$ then

$$Tr(xy) = Tr(yx), \quad Tr((xy)z) = T(x(yz)) \quad \text{and}$$

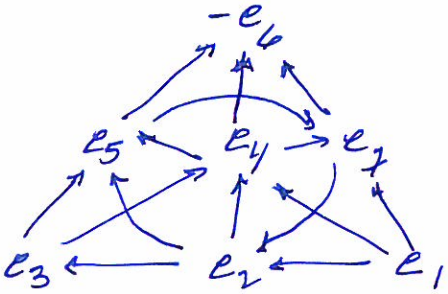
$$N(xy) = N(x)N(y).$$

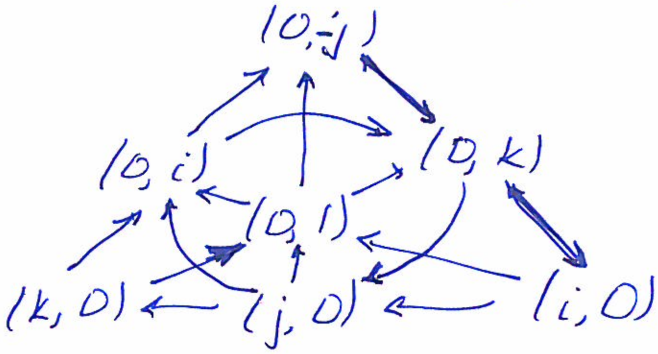
Since

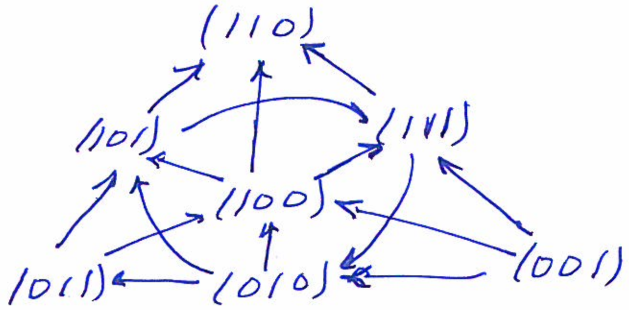
$$(x_1, 0)(x_2, 0) = (x_1 x_2, 0), \qquad (0, y_1)(0, y_2) = (-\bar{y}_2 y_1, 0)$$

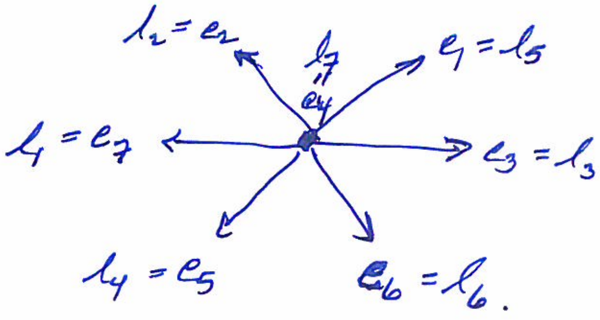
$$(x_1, 0)(0, y_2) = (0, y_2 x_1) \qquad (0, y_1)(x_2, 0) = (0, y_1 \bar{x}_2)$$

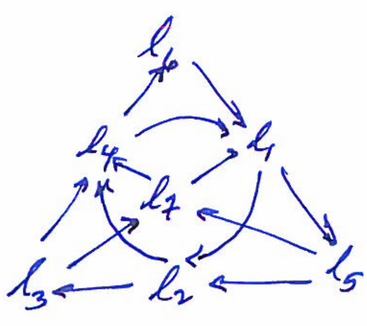
the following diagrams can serve as helpful mnemonics for the multiplication in $\mathbb{O}$,











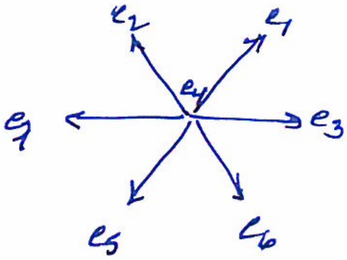
is the Pollack indexing.

②

Notes and References

The octonions are treated in [Bou, Alg, Ch. III, App. No 3].
The solution to the HW is found in [Bou, Alg. Ch III, App. No. 2 Prop 2 (iii)]. The choice of $e_0, \dots, e_7$ here follows [Bou, Alg, Ch. III App. No. 3].

The 7dimensional representation of G_2



$$e_{\alpha_1} = \begin{pmatrix} 0 & 1 & & & & & \\ & 0 & 0 & & & & \\ & & 0 & -2 & & & \\ & & & 0 & 1 & & \\ & & & & 0 & 0 & \\ & & & & & 0 & 1 \\ & & & & & & 0 \end{pmatrix} \qquad x_{\alpha_1}(c) = \begin{pmatrix} 1 & c & & & & & \\ & 1 & 0 & & & & \\ & & 1 & -2c & -c^2 & & \\ & & & 1 & c & & \\ & & & & 1 & & \\ & & & & & 1 & c \\ & & & & & & 1 \end{pmatrix}$$

$$e_{\alpha_2} = \begin{pmatrix} 0 & 0 & & & & & \\ & 0 & 1 & & & & \\ & & 0 & 0 & & & \\ & & & 0 & 0 & & \\ & & & & 0 & 1 & \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{pmatrix} \qquad x_{\alpha_2}(c) = \begin{pmatrix} 1 & 0 & & & & & \\ & 1 & c & & & & \\ & & 1 & 0 & & & \\ & & & 1 & 0 & & \\ & & & & 1 & c & \\ & & & & & 1 & 0 \\ & & & & & & 1 \end{pmatrix}$$

$$f_{\alpha_1} = \begin{pmatrix} 0 & & & & & & \\ 1 & 0 & & & & & \\ & 0 & 0 & & & & \\ & & -1 & 0 & & & \\ & & & 2 & 0 & & \\ & & & & 0 & 0 & \\ & & & & & 1 & 0 \end{pmatrix} \qquad x_{-\alpha_1}(c) = \begin{pmatrix} 1 & & & & & & \\ c & 1 & & & & & \\ & 0 & 1 & & & & \\ & & -c & 1 & & & \\ & & -c^2 & 2c & 1 & & \\ & & & & 0 & 1 & \\ & & & & & c & 1 \end{pmatrix}$$

$$f_{\alpha_2} = \begin{pmatrix} 0 & & & & & & \\ 0 & 0 & & & & & \\ & 1 & 0 & & & & \\ & & 0 & 0 & & & \\ & & & 0 & 0 & & \\ & & & & 1 & 0 & \\ & & & & & 0 & 0 \end{pmatrix} \qquad x_{-\alpha_2}(c) = \begin{pmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ & c & 1 & & & & \\ & & 0 & 1 & & & \\ & & & 0 & 1 & & \\ & & & & c & 1 & \\ & & & & & 0 & 1 \end{pmatrix}$$

$$n_1(c) = x_{\alpha_1}(c)\, x_{-\alpha_1}(-c^{-1})\, x_{\alpha_1}(c) = h_{\alpha_1^\vee}(c)\, n_1$$

$$= \begin{pmatrix} 0 & c & & & & & \\ -c^{-1} & 0 & & & & & \\ & & 0 & 0 & -c^2 & & \\ & & 0 & -1 & 0 & & \\ & & -c^{-2} & 0 & 0 & & \\ & & & & & 0 & c \\ & & & & & -c^{-1} & 0 \end{pmatrix}$$

$$h_{\alpha_1^\vee}(c) = \begin{pmatrix} c & & & & & & \\ & c^{-1} & & & & & \\ & & c^2 & & & & \\ & & & 1 & & & \\ & & & & c^{-2} & & \\ & & & & & c & \\ & & & & & & c^{-1} \end{pmatrix} \qquad n_1 = \begin{pmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & 0 & -1 & & \\ & & 0 & -1 & 0 & & \\ & & -1 & 0 & 0 & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{pmatrix}$$

$$n_2(c) = x_{\alpha_2}(c)\, x_{-\alpha_2}(-c^{-1})\, x_{\alpha_1}(c) = h_{\alpha_2^\vee}(c)\, n_2 \quad \text{with}$$

$$h_{\alpha_2^\vee}(c) = \begin{pmatrix} 1 & & & & & & \\ & c & & & & & \\ & & c^{-1} & & & & \\ & & & 1 & & & \\ & & & & c & & \\ & & & & & c^{-1} & \\ & & & & & & 1 \end{pmatrix} \qquad n_2 = \begin{pmatrix} 1 & & & & & & \\ & 0 & 1 & & & & \\ & -1 & 0 & & & & \\ & & & 1 & & & \\ & & & & 0 & 1 & \\ & & & & -1 & 0 & \\ & & & & & & 1 \end{pmatrix}$$

Bourbaki Lie Ch. 9 §3 Ex 7

Let $\mathbb{O} = \mathbb{R}\text{-span}\{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$

$V = \mathbb{R}\text{-span}\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$

$E = \mathbb{R}\text{-span}\{e_1, e_2, e_3, e_5, e_6, e_7\}$

$\mathbb{C} = \mathbb{R}\text{-span}\{e_0, e_4\}$

Let

$$G = \mathrm{Aut}_{\mathbb{R}\text{-alg}}(\mathbb{O}) \quad \text{and} \quad H = \mathrm{Stab}_G(e_4)$$

The quadratic form

$$\begin{matrix} Q : V \to \mathbb{R} \\ v \mapsto N(v) \end{matrix}$$ has orthonormal basis $\{e_1, \ldots, e_7\}$.

The Hermitian form

$\Phi : E \to \mathbb{C}$ has orthonormal basis $\{e_1, e_2, e_3\}$

Then

$$\begin{matrix} G \hookrightarrow SO(Q) \cong SO_7(\mathbb{R}) \\ g \mapsto g|_V \end{matrix} \quad \text{and} \quad \begin{matrix} H \xrightarrow{\sim} SU(\Phi) = SU_3(\mathbb{C}) \\ h \mapsto h|_E \end{matrix}$$

$$\begin{matrix} G/H \xrightarrow{\sim} S^6 \\ g \mapsto g(e_4) \end{matrix} \quad \text{and} \quad G/H \cong S^6.$$

where $S^6 = \{v \in V \mid N(v) = 1\}$ is the sphere in V. $(\dim_{\mathbb{R}}(V) = 7)$

Let $T \subseteq H$ be given by

$$T = \left\{ t \in G \;\middle|\; \begin{array}{ll} t(e_0) = e_0, & t(e_4) = e_4 \\ t(e_1) = \alpha e_1, & t(e_5) = \bar{\alpha} e_5 \\ t(e_2) = \beta e_2, & t(e_6) = \bar{\beta} e_6 \\ t(e_3) = \gamma e_3, & t(e_7) = \bar{\gamma} e_7 \end{array} \quad \begin{array}{l} \text{with } \alpha, \beta, \gamma \in \mathbb{C} \\ |\alpha| = |\beta| = |\gamma| = 1 \\ \alpha\beta\gamma = 1 \end{array} \right\}$$

This is a torus in $H \cong SU_3(\mathbb{C})$.

Note that $\dim(H) = \dim(SU_3(\mathbb{C})) = 8$

$$\begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$$

$\dim(G/H) = 6$

So $\dim(G) = 14$.

Then let

$N = N_G(T) =$ normalizer of T in G

N stabilizes $\{\pm e_1, \pm e_2, \pm e_3, \pm e_4, \pm e_5, \pm e_6, \pm e_7\}$ and N/T is order 12.

Computing the order of $G_2(\mathbb{F}_2)$

Let $G = G_2(\mathbb{F}_2)$.

Since $W_0 = \langle s_1, s_2 \mid s_i^2 = 1, (s_1 s_2)^6 = 1 \rangle$ then $\mathrm{Card}(W_0) = 12$

$$G/B = \bigsqcup_{w \in W_0} BwB \quad \text{with} \quad BwB \simeq \mathbb{F}_2^{\ell(w)}$$

So
$$\mathrm{Card}(G/B) = \sum_{w \in W_0} 2^{\ell(w)} = P_{W_0}(t)\Big|_{t=2}$$

$$= \frac{(1-t^2)(1-t^6)}{(1-t)^2}\Bigg|_{t=2} = \frac{(1-4)(1-2^6)}{(-1)^2} = 3\cdot(2^6-1)$$

$$= 3 \cdot 63 = 3 \cdot 21 \cdot 3 = 3^3 \cdot 7.$$

Then
$$B = \left\{ h_{\alpha_1^\vee}(d_1)\, h_{\alpha_2^\vee}(d_2)\, x_{\alpha_1}(c_1)\, x_{\alpha_2}(c_2) \cdots x_{3\alpha_1+2\alpha_2}(c_6) \;\middle|\; \begin{array}{l} d_1, d_2 \in \mathbb{F}_2^\times \\ c_1, \ldots, c_6 \in \mathbb{F}_2 \end{array} \right\}$$

So $\mathrm{Card}(B) = 1 \cdot 1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^6$.

So
$$\mathrm{Card}(G) = \mathrm{Card}(G/B) \cdot \mathrm{Card}(B)$$
$$= 3^3 \cdot 7 \cdot 2^6 = 12096.$$

Finding $G_2(\mathbb{Z})$

Let

$$\mathbb{O}_{\mathbb{Z}} = \mathbb{Z}\text{-span}\left\{ e_1, e_2, e_3, e_4, e_5, e_6, e_7, \frac{e_1+e_2+e_3+e_5}{2}, \frac{e_0+e_1+e_2+e_4}{2}, \frac{e_0+e_1+e_3+e_7}{2}, \frac{e_0+e_1+e_5+e_6}{2} \right\}$$

Let $\varphi \in \mathrm{Aut}(\mathbb{O}_{\mathbb{Z}})$. Then φ is determined by

$$\varphi(e_1), \varphi(e_2), \varphi(e_3) \in \mathbb{O}_{\mathbb{Z}}$$

and
$$N(\varphi(e_1)) = N(\varphi(e_2)) = N(\varphi(e_3)) = 1$$
$$\mathrm{Tr}(\varphi(e_1)) = \mathrm{Tr}(\varphi(e_2)) = \mathrm{Tr}(\varphi(e_3)) = 0.$$

There $\mathrm{Card}(\{x \in \mathbb{O}_{\mathbb{Z}} \mid N(x)=1, \mathrm{Tr}(x)=0\}) = 126$

With respect to the basis $e_1, e_2, \ldots, e_7$ these have entries in $0, \pm 1, \pm\frac{1}{2}$.

Also $G_2(\mathbb{Z}) \cong G_2(\mathbb{F}_2)$ and $\mathrm{Card}(G_2(\mathbb{F}_2)) = 2^6 \cdot 3^3 \cdot 7 = 12096$.

Computing $G(\mathbb{Z}) \backslash G(\mathbb{F}_p)/P_p$.

These are $G(\mathbb{Z})$ orbits on

$$G(\mathbb{F}_p)/P_p = \left\{ \begin{matrix} \text{lines, or planes, or flags} \\ \text{in } \mathbb{O}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{F}_p \end{matrix} \right\}$$

A <u>null line</u> is $V_1 = \mathrm{span}\{v\}$ with $v \in V$ and $N(v)=0$.

A <u>null plane</u> is $V_2 = \mathrm{span}\{v_1, v_2\}$ with $xy=0$ for $x, y \in V_2$

Here $V = \mathrm{span}\{e_1, e_2, \ldots, e_7\}$.