Flags, Cohomology and Positivity

MUMS talk
3 August 2012

$$1\otimes D_{\alpha_1} = \begin{array}{ccc} & -\alpha_1 & \\ \alpha_1 & & -(\alpha_1+\alpha_2) \\ \alpha_1+\alpha_2 & & -\alpha_2 \\ & \alpha_2 & \end{array}$$

$$1\otimes D_{-\alpha_2} = \begin{array}{ccc} & -\alpha_2 & \\ -(\alpha_1+\alpha_2) & & \alpha_2 \\ -\alpha_1 & & \alpha_1+\alpha_2 \\ & \alpha_1 & \end{array}$$

$$Z_{pt} = \begin{array}{ccc} & (-\alpha_1)(-\alpha_2)(-(\alpha_1+\alpha_2)) & \\ 0 & & 0 \\ 0 & & 0 \\ & 0 & \end{array}$$

Vertices are polynomials in α_1, α_2

t_{s_2} flips

$f_1 \leftrightarrow f_{s_2}$

$f_{s_1} \leftrightarrow f_{s_2 s_1}$

$f_{s_1 s_2} \leftrightarrow f_{s_1 s_2 s_1}$

t_{s_1} flips

$f_1 \leftrightarrow f_{s_1}$

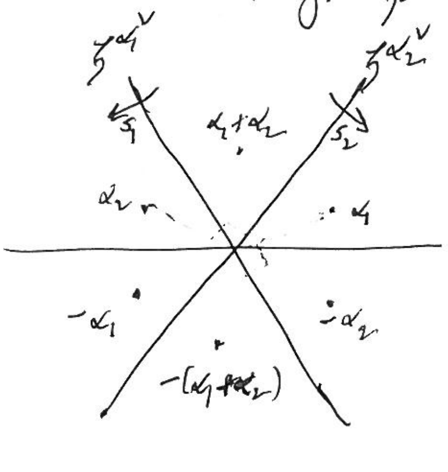
$f_{s_2} \leftrightarrow f_{s_1 s_2}$

$f_{s_2 s_1} \leftrightarrow f_{s_1 s_2 s_1}$

The BGG operators are

$$A_1 = (1+t_{s_1})\frac{1}{1\otimes D_{-\alpha_1}} \qquad A_2 = (1+t_{s_2})\cdot\frac{1}{1\otimes D_{-\alpha_2}}$$

coming from a dihedral group of order 6



$$Z_1 = A_1 Z_{pt} = \begin{array}{ccc} & (-\alpha_2)(-(\alpha_1+\alpha_2)) & \\ (-\alpha_2)(-(\alpha_1+\alpha_2)) & & 0 \\ 0 & & 0 \\ & 0 & \end{array} \qquad Z_2 = A_2 Z_p t \quad \begin{array}{ccc} & \alpha_1(\alpha_1+\alpha_2) & \\ 0 & & \alpha_1(\alpha_1+\alpha_2) \\ 0 & & 0 \\ & 0 & \end{array}$$

$$Z_{12} = A_1 A_2 Z_{pt} = \begin{array}{ccc} & -(\alpha_1+\alpha_2) & \\ -(\alpha_1+\alpha_2) & & -\alpha_1 \\ -\alpha_1 & & 0 \\ & 0 & \end{array} \qquad Z_{21} = A_2 A_1 Z_{pt} = \begin{array}{ccc} & -(\alpha_1+\alpha_2) & \\ -\alpha_2 & & -(\alpha_1+\alpha_2) \\ 0 & & -\alpha_2 \\ & 0 & \end{array}$$

and $$Z_{121} = A_2 A_1 A_2 Z_{pt} = A_1 A_2 A_1 Z_{pt} = \begin{array}{ccc} & 1 & \\ 1 & & 1 \\ 1 & & 1 \\ & 1 & \end{array}$$

Multiplication is <u>pointwise</u>:

$$Z_{12} Z_{21} = \begin{array}{ccc} & -(\alpha_1+\alpha_2)^2 & \\ \alpha_2(\alpha_1+\alpha_2) & & \alpha_1(\alpha_1+\alpha_2) \\ 0 & & 0 \\ & 0 & \end{array} = \cancel{(\alpha_1+\alpha_2)}$$

$$= \begin{array}{ccc} & \alpha_2(\alpha_1+\alpha_2) & \\ \alpha_2(\alpha_1+\alpha_2) & & 0 \\ 0 & & 0 \\ & 0 & \end{array} + \begin{array}{ccc} & \alpha_1(\alpha_1+\alpha_2) & \\ 0 & & \alpha_1(\alpha_1+\alpha_2) \\ 0 & & 0 \\ & 0 & \end{array} = Z_1 + Z_2$$

<u>Problem</u> Find c_{uv}^{w} given by $Z_u Z_v = \sum_w' c_{uv}^{w} Z_w$

<u>Smaller Problem</u> Show that $c_{uv}^{w} \in \mathbb{Z}_{\geq 0}[-\alpha_1, -\alpha_2]$.

Flags

A *flag* is an element of

$$Fl = \left\{ \vec{V} = (0 \subseteq V_1 \subseteq \cdots \subseteq V_n = \mathbb{C}^n) \;\middle|\; \begin{array}{l} V_i \text{ is a subspace of } \mathbb{C}^n \\ \dim V_i = i \end{array} \right\}$$

$$= \left\{ \begin{pmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_n \\ | & | & & | \end{pmatrix} B \;\middle|\; V_i = \operatorname{span}\{b_1, \ldots, b_i\} \right\}$$

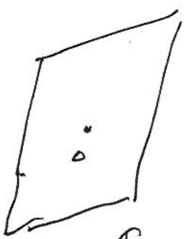
$$= GL_n / B, \quad \text{where } B = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \subseteq GL_n(\mathbb{C})$$

Example $n = 2$. We want $0 \subseteq V_1 \subseteq \mathbb{C}^2$

and $V_1 = \operatorname{span}\left\{ \begin{pmatrix} a \\ b \end{pmatrix} \right\} = \operatorname{span}\left\{ \begin{pmatrix} c \\ 1 \end{pmatrix} \right\}$ or $V_1 = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$

So $Fl = \left\{ \begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix} B \;\middle|\; c \in \mathbb{C} \right\} \sqcup \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} B \right\}$

$=$



$\sqcup \; \bullet$

$\mathbb{C}$

But

$$\begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix} B = \begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c^{-1} & 1 \\ 0 & -c \end{pmatrix} B = \begin{pmatrix} 1 & 0 \\ c^{-1} & 1 \end{pmatrix} B \quad \text{if } c \neq 0$$

So $Fl = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B \right\} \sqcup \left\{ \begin{pmatrix} 1 & 0 \\ c^{-1} & 1 \end{pmatrix} B \;\middle|\; c^{-1} \in \mathbb{C} \right\}$

$\bullet \; \sqcup$

$\mathbb{C}$

As $c^{-1} \to 0$ then $\begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix} B$ gets close to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} B$.

<u>Example</u> $n=3$: GL_n/B. Let $x_1(c) = \begin{pmatrix} 1 & c & \\ & 1 & \\ & & 1 \end{pmatrix}$ $\quad s_1 = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix}$

$$X_{pt} = B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} B$$

$$X_1 = \{ x_1(c) s_1 B \mid c \in \mathbb{C} \} = \left\{ \begin{pmatrix} c & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} B \;\middle|\; c \in \mathbb{C} \right\}$$

$$X_2 = \{ x_2(c) s_2 B \mid c \in \mathbb{C} \} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & 1 \\ 0 & 1 & 0 \end{pmatrix} B \;\middle|\; c \in \mathbb{C} \right\}$$

$$X_{12} = \{ x_1(c_1) s_1 x_2(c_2) s_2 B \mid c_1, c_2 \in \mathbb{C} \} = \left\{ \begin{pmatrix} c_1 & c_2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} B \;\middle|\; c_1, c_2 \in \mathbb{C} \right\}$$

$$X_{21} = \{ x_2(c_1) s_2 x_1(c_2) s_1 B \mid c_1, c_2 \in \mathbb{C} \} = \left\{ \begin{pmatrix} c_2 & 1 & 0 \\ c_1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} B \;\middle|\; c_1, c_2 \in \mathbb{C} \right\}$$

$$X_{121} = \{ x_1(c_1) s_1 x_2(c_2) s_2 x_1(c_3) s_1 B \mid c_1, c_2, c_3 \in \mathbb{C} \} = \left\{ \begin{pmatrix} c_3 c_2 + c_1 & c_3 & 1 \\ c_2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} B \;\middle|\; c_1, c_2, c_3 \in \mathbb{C} \right\}$$

Then, the <u>Schubert varieties</u> are

$$\overline{X_{pt}},\ \overline{X_1},\ \overline{X_2},\ \overline{X_{12}},\ \overline{X_{21}},\ \overline{X_{121}}.$$

Modification

Old: $D_\lambda + D_\mu = D_{\lambda+\mu}$ and $D_{-\lambda} = -D_\lambda$.

New

$$D_{\lambda+\mu} = D_\lambda + D_\mu - D_\lambda D_\mu.$$

Newer

$$D_{\lambda+\mu} = D_\lambda + D_\mu + a_{11} D_\lambda D_\mu + a_{21} D_\lambda^2 D_\mu + a_{12} D_\lambda D_\mu^2 + \cdots$$